Karlsruhe Institute of Technology

Institute of Algebra and Geometry
Englerstr. 2
D-76131 Karlsruhe

Phone: +49 721 608-43800
Web: http://www.math.kit.edu/iag

Bachelor Thesis

Models for Synthetic Differential Geometry

by

Julian Bitterwolf

Bachelor Thesis

Models for Synthetic Differential Geometry

Julian Bitterwolf

13 November 2017

Institute of Algebra and Geometry
at Karlsruhe Institute for Technology (KIT)

Abstract. Synthetic differential geometry provides an axiomatic definition of differential calculus by introducing infinitesimals in the form of non-zero numbers that square to zero. In the classical logic of sets, trying to implement a number line with such nilsquare infinitesimal elements quickly leads to inconsistencies. However, synthetic differential geometry can exist within topoi different from the topos of sets. These categories have an internal constructive logic, a system which comes without the law of excluded middle. This thesis shows the construction of line objects in a topos of presheaves of affine schemes as well as in the smooth Zariski topos which is built on smooth algebras.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Bachelor Thesis selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe. Die wörtlich oder inhaltlich übernommenen Stellen habe ich als solche kenntlich gemacht. Die Regeln des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der Fassung vom Mai 2010 habe ich beachtet.

Karlsruhe, den 13. November 2017

Julian Bitterwolf

Acknowledgements

I have learned a lot during the work on my bachelor project and would like to thank those people who made is possible and helped me along the way.

First, I would like to thank Prof. Dr. Frank Herrlich for letting me work at a project in his group, supporting and guiding me though it.

I worked closely with Dr. Felix Wellen, who usually was my first contact person and my supervisor. Thanks for everything, Felix, this would not have been possible without your knowledge, support and time.

A special thanks goes to a dear friend of mine, Jakob von Raumer, for the many helpful discussions, his help correcting the final thesis, and general advice.

I would also like to thank my parents and friends for their support during this project, and my beloved girlfriend for her stylistic advice and all the tasty cakes.

Contents

1 Ring Objects

A basic concept of geometry is the line. While in classical differential geometry the line is represented by the real numbers, synthetic approaches use other objects that should still incorporate some features of the number line. Most importantly, we want to be able to perform basic arithmetic operations on the line, namely addition, subtraction and multiplication. Inside an arbitrary category $\boldsymbol{C}$ with finite products (including a terminal object 1), an object with those arithmetic operations is described by the notion of a ring object.

1.1 Arithmetic operations

Definition 1. A *ring object* consists of the following data (For our purposes, we want all ring objects to be commutative and unitary.):

- an object R
- an 'addition' map $\alpha : R \times R \to R$ which is associative and commutative in the sense that the following two diagrams commute:

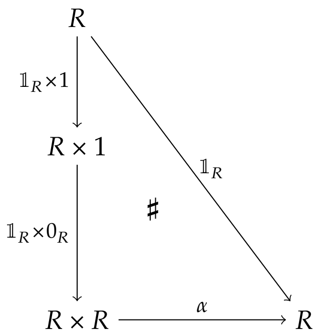
(1.1)

$$\begin{array}{ccc} R \times R & & \\ {\scriptstyle (\pi_2, \pi_1)} \downarrow & \searrow^{\alpha} & \\ & & R \\ & \nearrow_{\alpha} & \\ R \times R & & \end{array} \qquad\qquad \begin{array}{ccc} R \times R \times R & \xrightarrow{\mathbb{1}_R \times \alpha} & R \times R \\ {\scriptstyle \alpha \times \mathbb{1}_R} \downarrow & & \downarrow {\scriptstyle \alpha} \\ R \times R & \xrightarrow{\alpha} & R \end{array}$$

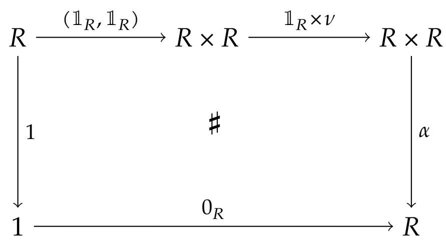
- an 'additive neutral' or 'zero' map $0_R : 1 \to R$ and an 'additive inverse' or 'negation' map ν :

$R \to R$, satisfying

(1.2)

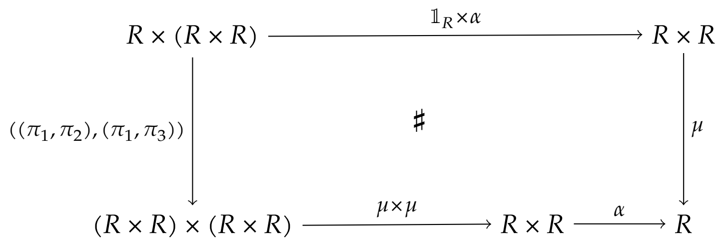


and



- a 'multiplication' map $\mu : R \times R \to R$, which is also associative and commutative: the diagrams 1.1 with α replaced by μ need to commute
- a 'multiplicative neutral' or 'one' map $1_R : 1 \to R$ such that the commutativity of the left diagram in 1.2 with α and 0 replaced by μ and 1_R, respectively, holds
- the distributivity law holds:

(1.3)



Later on, we will use an indirect way to specify a ring object: Instead of giving the arithmetics as morphisms in $\boldsymbol{C}$, one achieves the same by defining a *coring object* in the opposite category $\boldsymbol{C}^{op}$. Naturally, such a coring object is an object $\overline{R}$ with all the arrows of the maps and diagrams above reversed.

Technically, the categorical product $\times$ is not associative in the strict sense: there is a natural isomorphism called associator $a_{ABC} : (A \times B) \times C \xrightarrow{\sim} A \times (B \times C)$ which is not necessarily the identity. Therefore, in the associativity diagram on the right hand side of 1.1, the formally correct practice would be to write $(R \times R) \times R$ instead of $R \times R \times R$ and $(\mathbb{1}_R \times \alpha) \circ a_{RRR}$ instead of $\mathbb{1}_R \times \alpha$. However, here and in what follows we ignore these technicalities and act as if those natural isomorphism that come directly from limits and colimits were identities. We think that inserting the associators in the relevant formulas should in all cases not change the results.

The composition of the diagonal $(\mathbb{1}_R, \mathbb{1}_R) : R \to R \times R$ with μ gives us the squaring operation $-^2$: $R \to R$. We define the subobject of nilsquare elements D to be the ring theoretic kernel of this squaring operation, which is the following equalizer:

$$D \xhookrightarrow{\text{eq}} R \underset{-^2}{\overset{0}{\rightrightarrows}} R \tag{1.4}$$

where $0 : R \to R$ stands for $0_R \circ 1$.

1.2 Element notation

While all of the above works perfectly fine for any category $\mathbf{C}$ with the necessary products, the categories in which we will eventually need them to establish a theory of synthetic differential geometry will be topoi and as such, they will have nice set-like properties. This allows us to use the convenient notation with curly brackets and element glyphs. This is explained in detail in [Mac Lane and Moerdijk, 1992, VI.6] and [Moerdijk and Reyes, 1991, Appx. 1.3]. For example, in a topos, the equalizer notation of diagram 1.4 can be abbreviated as

$$D := \{d \in R \mid d^2 = 0\}. \tag{1.5}$$

The ring objects in the case that $\mathbf{C}$ is the category of sets are exactly the rings from commutative algebra; to avoid confusion, we call them set-rings. In the category of topological spaces, we get topological rings as ring objects. Here, ring operations are continuous maps, and thus also all maps constructed by concatenating them, i.e. all maps that are given by a polynomial. An example is $\mathbb{R}$ with the usual topology where open intervals form a basis of open subsets.

Since we work in a general category $\mathbf{C}$, there is no immediate sense in the talking about an *element* of an object A in $\mathbf{C}$. However, the notion of elements can be expanded: a generalized element of A is a morphism with codomain A. For an object X of $\mathbf{C}$, the elements of $\mathbf{C}(X, A)$ are called generalized elements of A *at stage* X, and those at stage 1 are called global elements. Generalized elements are useful because of Yoneda's lemma and embedding: specifying a morphism from A to B

is equivalent to specifying a natural transformation from $\mathbf{C}(-,A)$ to $\mathbf{C}(-,B)$, i.e. giving a morphism $\mathbf{C}(X,A) \to \mathbf{C}(X,B)$ natural in X. For a ring object R, the set $\mathbf{C}(X,R)$ is a set-ring: addition is defined as $\alpha_X : \begin{array}{ccc} \mathbf{C}(X,R) \times \mathbf{C}(X,R) & \longrightarrow & \mathbf{C}(X,R) \\ (f,g) & \longmapsto & \alpha \circ (f,g) \end{array}$ and the other ring operations work analogously. This allows us to use symbolic notation for ring maps as if we would define them by writing terms that depend on elements. For example, a naive construction like $x \mapsto x^4 - x + 1$ actually describes a morphism $R \to R$. Later on, we will use this naive but working description of ring maps to avoid lengthy formulas with many ring operation and projection maps. For a more in-depth discussion of generalized elements, refer to [Kock, 2010, Appx. 9.2] or [Kock, 2006, ch. II.1].

2 The Axiomatisation of Synthetic Differential Geometry

2.1 The idea

The idea of synthetic differential geometry is to define derivatives without using the limit process that relies on the convenient but sometimes impalpable properties of $\mathbb{R}$.

The idea that we will pursue can be established in an example: How do we calculate the derivative of a function like $f(x) = x^3 + 1$ at a point x_0 in high school mathematics? It is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 1 - x^3 - 1}{h}. \tag{2.1}$$

If we only take a look at the term $f(x+h) = x^3 + 3x^2h + 3xh^2 + h^3 + 1$ and separate the powers of h, then we see that it consists of the following parts:

The h^0-term $x^3 + 1 = f(x)$ will be subtracted for the calculation of the derivative.

The h^1-term $3x^2 \cdot h$ is just h times the derivative f' of f.

The h$^{\geq 2}$-terms $3xh^2 + h^3$ do not matter for determining the derivative.

That means that, at least in this case where f is a polynomial, what we do when calculating the derivative of f is just the same as calculating $f(x+h)$ and extracting its term linear in h.

Something similar happens in the Taylor expansion:

$$f(x_0 + d) = f(x_0) + f'(x_0)d + \frac{f''(x_0)}{2}d^2 + \frac{f'''(x_0)}{6}d^3 + \ldots \tag{2.2}$$

The first derivative is the part of the expansion that is linear in d. A way to formulate this is that calculating the derivative of a function is basically calculating $f(x_0 + d) - f(x_0)$ 'modulo' powers of d of order at least 2. Unfortunately, the real numbers don't contain such a number d that squares to zero, and forming a ring quotient modulo some finite – and thus invertible – number only yields the 0 ring.

2.2 The axiom

This difficulty with $\mathbb{R}$ cannot be circumvened by taking another set-ring.
Non-standard analysis uses approaches like the hyperreal numbers that introduce infinitesimals, but those are not nilsquare. The idea of synthetic differential geometry is to use a ring object in an appropriate category. The formulation of what we want from that ring object to make it a useful starting point for synthetic differential geometry was given by Kock and Lavwere with the following axiom:

Axiom 1
There is a non-zero ring R that can be exponentiated by its subobject of nilsquare elements D and the map

$$\chi\colon \begin{array}{ccc} R \times R & \longrightarrow & R^D \\ (a, b) & \longmapsto & (d \mapsto a + b \cdot d) \end{array} \tag{2.3}$$

is an isomorphism.

The exponential object R^D is defined by the property that for every other object A, there is a natural bijection

$$\lambda : \mathbf{C}(A \times D, R) \cong \mathbf{C}(A, R^D)\ , \tag{2.4}$$

called currying. The exponential object is a kind of internal version of the *set* $\mathbf{C}(D, R)$ of morphisms from D to R. For instance, this set is equal to the set of global elements of the exponenial object: $\mathbf{C}(1, R^D) = \mathbf{C}(1 \times D, R)$. Also, in the familiar category ***Set***, the set of functions fulfills the universal property of the exponential object. On all accounts, regarding the exponential object as such an 'object of morphisms' makes it easier to talk about its elements and lets us write down a map into it like we did in the axiom. It might nevertheless allow the axiom to make sense in a more general scope to define the map χ explicitly as

$$\chi = \lambda(\alpha \circ (\pi_1, \mu \circ (\pi_2, \iota_D \circ \pi_3)) . \tag{2.5}$$

In the above discussed symbolic notation of generalized elements, this reads as taking $A = R \times R$ and applying λ to $(a, b, d) \mapsto a + b \cdot d$. We keep this a bit more precise definition in mind, but proceed with the notation as in 2.3 and remember what it stands for.

The isomorphism χ from Axiom 1 read backwards says that for every map $g : D \longrightarrow R$, there exist unique $a, b \in R$ with $g = d \mapsto a + b \cdot d$. We use this to define the first derivative of a function:

Definition 2. For a function $f : R \longrightarrow R$ and $c \in R$, regard $g : D \longrightarrow R$ with $g(d) = f(c + d)$. Use axiom 1 to write g as $g(d) = g(0) + b \cdot d = f(c) + b \cdot d$. Define $f'(c) = b$.

2.3 Logical consequences

The reader might immediately notice that there were no conditions for f that make sure that it is differentiable beforehand: All functions are differentiable. As this clearly is not the case in 'usual' mathematics, the category $\mathbf{C}$ must have some properties that make it quite different from the categories of sets or of topological spaces. To be more precise with what we mean by 'usual' mathematics, we can think of everything that is built upon the ZFC axiom scheme or some extension of it. Here, functions between sets A and B are certain subsets of the set of pairs $A \times B$. These ZFC-sets and functions form the category ***Set***.

A very drastic hint that we have to abandon some properties of the classical mathematical world is the following statement:

Theorem 3. *Axiom 1 is inconsistent with classical logic: For a set-ring R, Axiom 1 implies $R = 0$, which it also explicitly prohibits.*

Proof: Define a function $c_0 : D \longrightarrow R$ as

$$c_0(d) = \begin{cases} 0 & \text{if } d = 0 \\ 1 & \text{if } d \neq 0 \end{cases} . \tag{2.6}$$

Axiom 1 yields a ring element b such that

$$c_0(d) = c_0(0) + b \cdot d = b \cdot d \ . \tag{2.7}$$

Claim. $D = \{0\}$.

Claim's proof: Assume the existence of a $d \in D$ with $d \neq 0$. Then 2.6 and 2.7 tell us that

$$1 = b \cdot d \ . \tag{2.8}$$

We multiply both sides by d to get the equation

$$d = 1 \cdot d = (b \cdot d) \cdot d = b \cdot d^2 = b \cdot 0 = 0 \ . \tag{2.9}$$

We conclude that there is no $d \in D - \{0\}$. □

Since any $b \in R$ satisfies $0 = c_0(0) = b \cdot 0 = 0$, the uniqueness condition from Axiom 1 allows R to only have one element, i.e. $R = \{0\}$, which contradicts the premise of Axiom 1. □

Within a consistent theory which includes Axiom 1, a function like c_0 in the proof of Theorem 3 should not exist. One way to guarantee this could consist of defining a class of 'nice' functions and having the set of morphisms $D \longrightarrow R$ only consist of those. However, this would need an a priori notion of differentiability and we wouldn't get anything from Axiom 1 that we wouldn't have without it. The other way is - by working in a different category from ***Set*** - to change our logical mindset in such a way that c_0 is not even a well-defined function. If we do not know for every element of D if it is either 0 or not 0, then writing down a computation rule that relies on that knowledge doesn't make sense.

3 Constructive Logic

3.1 Topoi different from *Set* have different internal logics

Usually, the logical rules that we use for deducing mathematical truths are fixed and we don't need to think about them. Logical propositions very much behave like sets, in the sense that given a set U of considered states and a predicate P on those states, we define $U_P := \{x \in U \mid P(x)\}$. Then for $x \in U$ we can express the proposition $P(x)$ as the set-theoretic statement $x \in U_P$. We therefore identify P with the set U_P. With this identification, logical connectives correspond to set operations. For example, logical conjunction is intersection

$$\{x \in U \mid P(x) \wedge Q(x)\} = U_{P\wedge Q} = U_P \cap U_Q = \{x \in U \mid x \in U_P \wedge x \in U_Q\}\,, \tag{3.1}$$

forming a negation is taking the complement

$$\{x \in U \mid \neg P(x)\} = U_{\neg P} = U - U_P = \{x \in U \mid x \notin U_P\} \tag{3.2}$$

and so on. Those set operations are limits and colimits in the category ***Set***, or more specifically in its subcategory of subsets of U. This inspires the idea to explore what happens if we do the same operations in a different category. Generally, it is established that this works well in what are called Heyting categories. The result is a 'constructive' or 'intuitionistic' logic, a generalization of classical logic. A great class of Heyting categories are the below-defined topoi. But for an easy example, take a topological space V (for example $\mathbb{R}$) and let the predicates be the open subsets of V (e.g. $(0,\infty) = \mathbb{R}_{x>0}$). Then logical conjunction works just in the same way as for sets, but the complement of an open set needn't be open itself ($\mathbb{R} - (0,\infty) = (\infty, 0]$ is not open). What we need to take as the negation of P instead is the interior of its complement:

$$V_{\neg P} = \operatorname{int}(V - V_P) = \bigcup_{A\cap V_P=\emptyset} A\ . \tag{3.3}$$

This immediately leads to the observation that the logical disjunction of a predicate and its negation is not necessarily always true: $V_P \cup V_{\neg P}$ might miss some elements on the boundary of V_P. As a concrete example take $V = \mathbb{R}$, $V_P = (0,\infty)$ (therefore P is 'being greater than zero'). Here, $\mathbb{R} - V_P = (\infty, 0]$ is not open. The negation is rather $V_{\neg P} = (\infty, 0)$. Thus, $V_P \cup V_{\neg P} = \mathbb{R} - 0$ is not all of $\mathbb{R}$.

This example of topological spaces is rather closely related to the category of sets, since ***Top*** is a concrete category where objects are sets with an additional structure and can be described in terms of their elements. A Heyting category has its *internal logic* where each object is a proposition, independently of its generalized (or otherwise understood) elements. Here, the logical operations of propositional calculus correspond to categorical constructions as presented in table 1.

category		internal logic	
product	$\times$	$\wedge$	and
coproduct	$\otimes$	$\vee$	or
terminal object	1	$\top$	True
initial object	$\emptyset$	$\bot$	False
exponential object	A^B	$B \Rightarrow A$	implication
exponetial of $\emptyset$	$\emptyset^A$	$\neg A$	negation

Table 1

3.2 The law of excluded middle

This circumstance that $V_P \cup V_{\neg P} = V$ does in general not hold is the most central peculiarity of constructive logic: the missing of the law of excluded middle. While in classical logic, for any proposition P, the proposition $P \vee \neg P$ is true, constructive logic relinquishes this inference. This does not mean that constructive logicians claim or assume that there is some proposition that is neither true nor false. In the $\mathbb{R}$-example above, negation of $V_P \cup V_{\neg P}$ is the empty interior of a single point. Having a concrete example of something being neither true nor false would even be inconsistent:

In classical and constructive logic alike, negation is defined as implication of falsehood:

$$\neg P \;:\equiv\; P \Rightarrow \bot\,. \tag{3.4}$$

If we had $\neg(P \vee \neg P)$, for some P, we could infer $\bot$ in the following way:

$$(\neg(P \vee \neg P)) \Rightarrow (\neg P \wedge \neg\neg P) \overset{3.4}{\equiv} (\neg P \wedge (\neg P \Rightarrow \bot)) \overset{\text{modus ponens}}{\Rightarrow} \bot\,. \tag{3.5}$$

Instead, when we do constructive mathematics we merely don't see $(P \vee \neg P)$ as a legitimate part of a proof.

Note that the $\equiv$ sign in 3.4 and 3.5 is an equal sign that we use if we talk about logical terms to the end that it doesn't get confused with an equality that might occur within a term.

3.3 Constructive proofs

A popular example where such an inconstructive proof is used is the following theorem from arithmetics:

Theorem 4. *There is a pair of irrational numbers a and b such that a^b is rational.*

An easy classical proof would be:

Classical Proof: Since at least 400 B.C. it is a well known (and by the way also constructively proven) fact that the square root of 2 is irrational. Use the law of excluded middle on the proposition '$\sqrt{2}^{\sqrt{2}}$ is rational' and its negation '$\sqrt{2}^{\sqrt{2}}$ is irrational' to split the proof into two cases:

Case '$\sqrt{2}^{\sqrt{2}}$ is rational': Take $a = \sqrt{2}, b = \sqrt{2}$.

Case '$\sqrt{2}^{\sqrt{2}}$ is irrational': Take $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$. Therewith

$$a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2 \in \mathbb{Q}\,. \tag{3.6}$$

□

We have proven the theorem, but we still don't know a single example of two such numbers that it predicts. Some people deem this unsatisfying and demand a concrete example for an existence proof.

Considering this example, it is possible, but a lot harder, to prove that $\sqrt{2}^{\sqrt{2}}$ is irrational (refer to the *Gelfond–Schneider theorem*) and therefore have $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$ witnessing the theorem. To demonstrate that we can easily proof theorem 4, instead of the quite complicated number $\sqrt{2}^{\sqrt{2}}$, take the binary logarithm of 3.

Lemma 5. $\log_2 3$ *is irrational.*

Proof: Assume $\log_2 3 \in \mathbb{Q}$. That means there are integers m, n with $n > 0$ and

$$\log_2 3 = \frac{m}{n}.$$

Exponentiating the number 2 by each side yields

$$3 = 2^{\frac{m}{n}}.$$

Taking the n^{th} power of both sides,

$$3^n = 2^m ,$$

makes it clear that

$$m = 0, n = 0$$

is the only integer solution, which is ruled out by $n > 0$. Therefore the assumption $\log_2 3 \in \mathbb{Q}$ leads to a contradiction. □

Note that using a proof by contradiction is problematic in constructive mathematics if it is used to show a positive statement, since double negation elimination ($\neg\neg P \Rightarrow P$) is logically equivalent to $P \vee \neg P$. Because of this, starting with "Assume $\neg P$" and deriving a contradiction from this is not a constructively valid proof of P. However, deriving a contradiction is exactly what we should do for proving a negation like $\log_2 3 \notin \mathbb{Q} \equiv \neg(\log_2 3 \in \mathbb{Q}) \equiv ((\log_2 3 \in \mathbb{Q}) \Rightarrow \bot)$.

Lemma 5 facilitates a short proof of Theorem 4:

Constructive proof of $\exists a, b \in \mathbb{R} - \mathbb{Q} : a^b \in \mathbb{Q}$:

Take $a = \sqrt{2}$, $b = \log_2 3$.

$$a^b = \sqrt{2}^{\log_2 3} = 2^{\log_2 3} = 3 \in \mathbb{Q} \tag{3.7}$$

□

4 An Algebraic Model based on the Affine Line

4.1 The plan

In this section, we construct the presheaf topos of a class of algebras and observe how Axiom 1 is incorporated inside that topos, following [Kock, 2010, Appx. 9.2]. By 'ring' we mean a commutative ring with 1, and ring morphisms are required to send the 1 of the domain to the codomain's. The recipe consists of the following steps:

- Understand the category ***k-Alg*** with its tensor product and polynomial rings.
- In $\mathbb{A}\boldsymbol{S_k} = \boldsymbol{k\text{-}Alg}^{op}$, regard the affine line $\mathbb{A} = k[X]^{op}$. It is a ring object.
- Show that $\mathbb{A}$ is finitely exponentiating: For an algebra W of vector space dimension n, $\mathbb{A}^{\overline{W}} = \mathbb{A}^n$.
- Via Yoneda embedding, interpret these notions in the presheaf topos of $\mathbb{A}\boldsymbol{S_k}$.
- Take $W = k[X]/(X^2)$ to get the nilsquare infinitesimals from Axiom 1.

4.2 Algebras and their tensor product

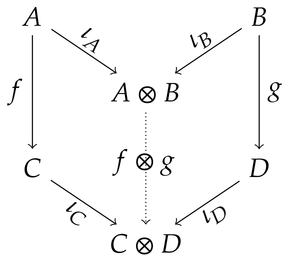
For a ring k, a k-algebra is a ring under k. That is a ring A together with a ring homomorphism $i_A : k \to A$; this structure morphism is most often omitted in notation. With this definition, a homomorphism f of algebras A and B is a ring homomorphism such that the following diagram commutes:

(4.1)
$$\begin{array}{ccc} & k & \\ {\scriptstyle i_A}\swarrow & & \searrow{\scriptstyle i_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

We denote the category of k-algebras by ***k-Alg***. This category has finite products per tuples with component-wise multiplication and addition, as well as finite coproducts per tensor product $\otimes$ of algebras. Since in general (see [MacLane, 1998, Thm. 1 in ch. V.4]), the contravariant Hom-functor sends colimits to limits , the tensor product has the property that $\boldsymbol{k\text{-}Alg}(A \otimes B, X) \underset{\text{in } A,B,X}{\overset{\text{natural}}{\cong}} \boldsymbol{k\text{-}Alg}(A, X) \times \boldsymbol{k\text{-}Alg}(B, X)$, and an element x of $A \otimes B$ can generally be written in the form $x = \sum_{i=1}^{n} a_i \otimes b_i$. The coproduct injections are

(4.2)
$$\iota_A : \begin{array}{ccc} A & \longrightarrow & A \otimes B \\ a & \longmapsto & a \otimes 1 \end{array} \quad \text{and} \quad \iota_B : \begin{array}{ccc} A & \longrightarrow & A \otimes B \\ b & \longmapsto & 1 \otimes b \end{array} .$$

If two maps of k-algebras $f : A \to C$ and $g : B \to D$ are given, then there is a canonical map $f \otimes g : A \otimes B \to C \otimes D$ due to the coproduct property of $A \otimes B$:



Consequently,

$$
\begin{aligned}
f \otimes g(a \otimes b) &= f \otimes g((a \otimes 1) \cdot (1 \otimes b)) \\
&= f \otimes g(\iota_A(a)) \cdot f \otimes g(\iota_B(b)) = \iota_C(f(a)) \cdot \iota_D(g(b)) \\
&= (f(a) \otimes 1) \cdot (1 \otimes g(b)) = f(a) \otimes g(b) \ .
\end{aligned}
\tag{4.3}
$$

4.3 A nice property of polynomial rings

The algebra $k[X]$, the polynomial ring over k in one variable, has the following universal mapping property: for an algebra A, any map $f : k[X] \to A$ is uniquely determined by $f(X) \in A$, and on the other hand, every $a \in A$ gives rise to a different map $ev_a : k[X] \to A$. Thus, we can identify elements of A with maps from $k[X]$ into A, and $k[X]$ is called the free k-algebra in one variable. Further, the polynomial ring of n variables $k[S_1, \dots, S_n]$ is free in n variables and is isomorphic to the n-fold tensor product of $k[X]$.

For what follows, we fix a k-algebra W which is finite-dimensional as a vector space over k and we choose a vector space basis $\mathcal{B} = (b_1, b_2, \dots, b_n)$ of W.

Lemma 6. *Any element x of $A \otimes W$ can be written in the form $x = \sum_{i=1}^{n} a_i \otimes b_i$ for some $a_1, \dots, a_n \in A$.*

Proof: Let $x = \sum_{j=1}^{m} c_j \otimes w_j$ for $c_j \in A, w_j \in W$. Write each w_j as a k-linear sum of the basis vectors to get

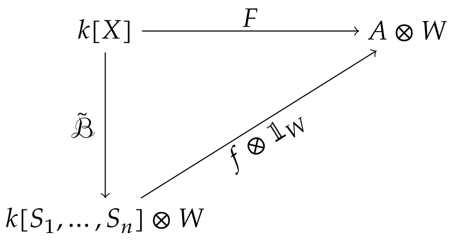
$$
x = \sum_{j=1}^{m} c_j \otimes \left(\sum_{i=1}^{n} \lambda ji \cdot b_i \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_j \otimes (\lambda ji \cdot b_i) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \lambda ji \cdot c_j \right) \otimes b_i. \tag{4.4}
$$

Take $a_i = \sum_{j=1}^{m} \lambda ji \cdot c_j$. □

Corresponding to the basis $\mathcal{B}$, we define the map $\tilde{\mathcal{B}} : \begin{array}{ccc} k[X] & \longrightarrow & k[S_1, \dots S_n] \otimes W \\ X & \longmapsto & \sum_{i=1}^{n} S_i \otimes b_i \end{array}$.

Lemma 7. *Let $F : k[X] \to A \otimes W$ be an arbitrary map of algebras, determined by $F(X) = \sum_{i=1}^{n} a_i \otimes \mathbb{b}_i$. Then there is a unique $f : k[S_1, \dots, S_n] \to A$ which makes the diagram*

(4.5)



commute. This f is evaluation at $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$, i.e. in a polynomial P, the map f replaces S_i with the i^{th} component of $F(X)$ with respect to the basis $\mathcal{B}$. Thus we have a bijection

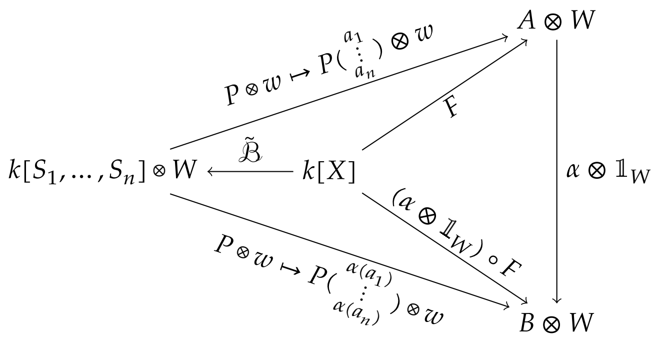
$$e_{\mathcal{B}} : \textbf{\textit{k-Alg}}(k[X], A \otimes W) \xrightarrow{\sim} \textbf{\textit{k-Alg}}(k[S_1, \dots, S_n], A) \ . \tag{4.6}$$

It is natural in A.

Proof: The fact that $k[S_1, \dots, S_n]$ is the free k-algebra in n variables means that the evaluation f is indeed a k-algebra homomorphism. To check commutativity of diagram 4.5, we calculate

$$\begin{aligned} ((f \otimes \mathbb{1}_W) \circ \tilde{\mathcal{B}})(X) &= f \otimes \mathbb{1}_W(\sum_{i=1}^{n} S_i \otimes \mathbb{b}_i) = \sum_{i=1}^{n} f(S_i) \otimes \mathbb{b}_i \\ &= \sum_{i=1}^{n} a_i \otimes \mathbb{b}_i = F(X) \ , \end{aligned} \tag{4.7}$$

which also makes the uniqueness apparent. For naturality, we have to check that for an arbitrary map $\alpha : A \to B$, the outer triangle of the following diagram commutes:



(4.8) .

This will be sufficient since those pure elements $P \otimes w$ generate the algebra $k[S_1, \dots, S_n] \otimes W$. It indeed commutes, since the algebra homomorphism α respects both addition and multiplication and hence can be pulled outside of the polynomial P. □

4.4 Opposites of algebras: affine schemes

We consider the opposite category $\mathbb{A}\boldsymbol{S_k} = \boldsymbol{k\text{-}Alg}^{op}$. It is the category of affine schemes over k which plays an important role in algebraic geometry. The scheme corresponding to the polynomial ring in one variable is called the affine line $\mathbb{A}^1$ or simply $\mathbb{A}$. We show that it indeed satisfies Axiom 1. Limits become colimits - and vice versa - if we switch to the opposite category. An overview of corresponding constructions is given in table 2.

k-algebras	$\boldsymbol{k\text{-}Alg}$	$\mathbb{A}\boldsymbol{S_k}$	aff. schemes over k
some object	A	$\overline{A}$	written with a bar
coproduct	$\otimes$	$\times$	product
final object	0	$\emptyset$	initial object
initial object	k	1	final object
pol. ring in 1 variable	$k[X]$	$\mathbb{A}$	affine line
its n-fold coproduct	$k[S_1, \dots, S_n]$	$\mathbb{A}^n$	its n-fold product

Table 2

The affine line becomes a ring object by defining the arithmetic operations of a coring in the opposite category:

$$\overline{\alpha}\colon \begin{array}{ccc} k[X] & \longrightarrow & k[A,B] \\ X & \longmapsto & A+B \end{array}, \tag{4.9}$$

$$\overline{\mu}\colon \begin{array}{ccc} k[X] & \longrightarrow & k[A,B] \\ X & \longmapsto & AB \end{array}, \tag{4.10}$$

$$\overline{0_{\mathbb{A}}}(X) = 0_k,\ \overline{\nu}(X) = -X,\ \overline{1_{\mathbb{A}}}(X) = 1_k\,. \tag{4.11}$$

Recognising that this indeed defines a coring object is straightforward; exemplarily, we check that the distributivity as in diagram 1.3 holds:

$$\begin{array}{ccccc} k[X,Y,Z] & & \xleftarrow[\displaystyle Y+Z \mapsfrom B]{\displaystyle X \mapsfrom A} & & k[A,B] \\ \Big\uparrow {\scriptstyle P\mapsto X,\ Q\mapsto Y,\ R\mapsto X,\ S\mapsto Z} & & & & \Big\uparrow {\scriptstyle T\mapsto AB} \\ k[P,Q,R,S] & \xleftarrow[\displaystyle RS \mapsfrom D]{\displaystyle PQ \mapsfrom C} & k[C,D] & \xleftarrow{\displaystyle C+D \mapsfrom T} & k[T] \end{array} \tag{4.12}$$

When going first up and then left in diagram 4.12, T is mapped to $X(Y+Z)$, while the left-left-up path maps T to $XY + XZ$. These are of course equal due to distributivity in $k[X,Y,Z]$.

Now we conclude that the affine line is exponentiating for any W as above: The diagram opposite to

4.5 is

$$\begin{array}{ccc} \mathbb{A} & \xleftarrow{\overline{F}} & \overline{A}\times\overline{W} \\ \uparrow{\scriptstyle \overline{\tilde{\mathcal{B}}}} & \swarrow{\scriptstyle \overline{f}\times \mathbb{1}_{\overline{W}}} & \\ \mathbb{A}^n\times\overline{W} & & \end{array} \quad , \tag{4.13}$$

and we observe that taking the statement opposite to Lemma 7 mutates 4.6 into natural bijection $e_{\mathcal{B}}$: $\mathbb{A}\boldsymbol{S_k}(\overline{A}\times\overline{W},\mathbb{A}) \xrightarrow{\sim} \mathbb{A}\boldsymbol{S_k}(\overline{A},\mathbb{A}^n)$. This means that $\mathbb{A}^n$ fulfills the definition of the exponential object 2.4; we write $\mathbb{A}^n \cong_{\mathcal{B}} \mathbb{A}^{\overline{W}}$, with the subscript $\mathcal{B}$ to remind us that the exact identification depends on the choice of vector space basis of W. If we interpret the exponential object as the set of morphisms from $\overline{W}$ to $\mathbb{A}$, the identification is concretely obtained by currying the map $\overline{\tilde{\mathcal{B}}}$:

$$\overline{\tilde{\mathcal{B}}}_1 \colon \begin{array}{ccc} \mathbb{A}^n & \xrightarrow{\sim} & \mathbb{A}\boldsymbol{S_k}(\overline{W},\mathbb{A}) \\ x & \longmapsto & \overline{\tilde{\mathcal{B}}}(x,-) \end{array} \quad . \tag{4.14}$$

For our logical and geometrical framework, we want to have the rich structure of a topos. An easy way to get a topos from a category is taking the topos of presheaves on it. This is works well if the underlying category is equivalent to a small. This is the case for category $\boldsymbol{k\text{-}Alg}_{fp}^{op}$ of opposites of finitely presented k-algebras, to which we restrict ourselves from now on. The presheaf topos of $\boldsymbol{k\text{-}Alg}_{fp}^{op}$ is $\boldsymbol{Set}^{(\boldsymbol{k\text{-}Alg}_{fp}^{op})^{op}} = \boldsymbol{Set}^{\boldsymbol{k\text{-}Alg}_{fp}}$. There is the Yoneda embedding $y\colon \begin{array}{ccc} \boldsymbol{k\text{-}Alg}_{fp}^{op} & \longrightarrow & \boldsymbol{Set}^{(\boldsymbol{k\text{-}Alg}_{fp}^{op})^{op}} \\ \overline{A} & \longmapsto & \boldsymbol{k\text{-}Alg}_{fp}^{op}(-,\overline{A}) \end{array}$ which is not only full and faithful, but also preserves existing limits and exponential objects. Those include the property of being a ring object. From the general

$$y(\overline{A}) = \boldsymbol{k\text{-}Alg}_{fp}^{op}(-,\overline{A}) = \boldsymbol{k\text{-}Alg}_{fp}(A,-)\ , \tag{4.15}$$

we see that in particular

$$y(1) = \boldsymbol{k\text{-}Alg}_{fp}(k,-) = (A \mapsto \{i_A\}) \tag{4.16}$$

is the functor that maps every object to the one-point set containing $i_A : k \to A$. Thus, it is indeed the terminal object in our presheaf category: $y(1) = 1$. We also take a look at $y(\mathbb{A}) = \boldsymbol{k\text{-}Alg}_{fp}(k[X],-)$. At an algebra A, it is the set $\textit{\textbf{k-Alg}}_{fp}(k[X], A) = \{ev_a \mid a \in A\}$, which is identical to A itself. Applying the functor $y(\mathbb{A})$ to a homomorphism $f : A \to B$ yields the map $ev_a \mapsto ev_{f(a)}$.

The next step is to understand what the presheaf belonging to the map $\overline{\tilde{\mathbb{B}}}_1$ does. Since the Yoneda embedding preserves limits and exponential objects, we can read the map as

$$y(\overline{\tilde{\mathbb{B}}}_1) : y(\mathbb{A})^n \to y(\mathbb{A})^{y(\overline{W})} . \tag{4.17}$$

We would to know what this map does to the elements of $y(\mathbb{A})^n$. Instead of having to regard all generalized elements $Nat(\phi, y(\mathbb{A})^n)$, we will still get the necessary information by only looking at the cases where ϕ is a representable presheaf. This means ϕ is of the form $y(\overline{A})$ for some $\overline{A} \in \textit{\textbf{k-Alg}}^{op}_{fp}$. That makes things a lot easier, since Yoneda's lemma tells us that $Nat(y(\overline{A}), y(\mathbb{A})^n) = \textit{\textbf{k-Alg}}^{op}_{fp}(A, \mathbb{A}^n)$. By definition of the opposite category, thisset is equal to $\textit{\textbf{k-Alg}}_{fp}(k[S_1, \dots, S_n], A)$. Its elements are the evaluation maps in n variables at $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ where the a_i are from A. Applying $y(\overline{\tilde{\mathbb{B}}}_1)$ to such a generalized element means applying its component at $\overline{A}$. Therefore, the result is a generalized element of $y(\mathbb{A})^{y(\overline{W})}$ at stage A. Using Yoneda's lemma again, this means it lies in the set $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \mathbb{A})^{\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \overline{W})} = \textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \mathbb{A}^{\overline{W}})$. This can be rewritten with the currying bijection λ to $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A} \times \overline{W}, \mathbb{A}) = \textit{\textbf{k-Alg}}_{fp}(k[X], A \otimes W)$. Chasing all the steps back through lemma 7, we can see that $y(\overline{\tilde{\mathbb{B}}}_1)$ at $\overline{A}$ is (the inverse of) the isomorphism $e_{\mathbb{B}}$ in 4.6. It sends the n-variable evaluation at $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ that would be f in lemma 7 to the one-variable evaluation $X \mapsto \sum_{i=1}^n a_i \otimes \mathscr{b}_i$ that would be F. So what is this map $X \mapsto \sum_{i=1}^n a_i \otimes \mathscr{b}_i$ when concretely written as a generalized element of $y(\mathbb{A})^{y(\overline{W})}$ at stage $y(\overline{A})$? Since we are in ***Set***, the exponential object $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \mathbb{A})^{\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \overline{W})}$ is actually the set of functions from $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \overline{W})$ to $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \mathbb{A})$. Elements of $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \overline{W})$ are generalized elements of $\overline{W}$ at stage $y(\overline{A})$, and they are homomorphisms from W to A. Given such a homomorphism d, we apply it to the tensor product of A with W to give us the retraction map

$$d^* : \begin{array}{ccc} A \otimes W & \longrightarrow & A \\ a \otimes w & \longmapsto & a \cdot d(w) \end{array} , \tag{4.18}$$

similar to the one for dual spaces in linear algebra. We postcompose d^* with the map $X \mapsto \sum_{i=1}^n a_i \otimes \mathscr{b}_i$ to get the map $X \mapsto \sum_{i=1}^n a_i \cdot d(\mathscr{b}_i)$, which an element of $\textit{\textbf{k-Alg}}^{op}_{fp}(\overline{A}, \mathbb{A})$, i.e. generalized element of $y(\mathbb{A})$ at stage $y(\overline{A})$. Taken all together and denoting generalized elements by element symbols as discussed in section 1, we started with an evaluation map defined by $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and applied the bijection $y(\overline{\tilde{\mathbb{B}}_1})$ to it to get a map that sends $d \in \overline{W}$ to the evaluation at $\sum_{i=1}^n a_i \cdot d(\mathbb{b}_i)$. Written out, this means the bijection is

$$y(\overline{\tilde{\mathbb{B}}_1}) \colon \begin{array}{ccc} y(\mathbb{A})^n & \xrightarrow{\sim} & y(\mathbb{A})^{y(\overline{W})} \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} & \longmapsto & d \mapsto \sum_{i=1}^n x_i \cdot d(\mathbb{b}_i) \end{array} . \tag{4.19}$$

The fact that this map is a bijection means that any element of $y(\mathbb{A})^{y(\overline{W})}$ is of this form. We express this insight in the following theorem:

Theorem 8. *For any map $f : y(\overline{W}) \to y(\mathbb{A})$, there are unique $a_1, \dots, a_n \in y(\mathbb{A})$, such that for all $d \in \overline{W}$ the following equation holds:*

$$f(d) = \sum_{i=1}^{n} a_i \cdot d(\mathbb{b}_i) . \tag{4.20}$$

We observe a resemblance with the pattern of Axiom 1. The affine line $y(\mathbb{A})$ will actually work out as the line object. Above, we already showed in which way it is a ring object.

To calculate $D := \{d \in y(\mathbb{A}) \mid d^2 = 0\}$, we write it as an equalizer:

$$D \xrightarrow{\text{eq}} R \underset{\_^2}{\overset{0}{\rightrightarrows}} R . \tag{4.21}$$

Since existing equalizers are preserved by $y(-)$, we can calculate it in $\boldsymbol{k\text{-}Alg}_{fp}^{op}$:

$$\overline{A} \xrightarrow{\text{eq}} \mathbb{A} \underset{\_^2}{\overset{0}{\rightrightarrows}} \mathbb{A} . \tag{4.22}$$

We can handle this in the opposite category, where the maps 0 and $\_^2$ correspond to $X \mapsto 0$ and $X \mapsto X^2$, respectively. Taking their coequalizer produces

$$k[X]/(X^2) \xleftarrow{X \mapsfrom X} k[X] \underset{X^2 \mapsfrom X}{\overset{0 \mapsfrom X}{\leftleftarrows}} k[X] \quad . \tag{4.23}$$

Consequently, we set $W = k[X]/(X^2)$, such that D becomes $y(\overline{W})$, and choose the vector space basis $\mathcal{B} = (1, X)$ of W. Putting this into Theorem 8 yields

Corollary 9. *Any map $f : D \to R$ can be uniquely written in the form*

$$f : d \mapsto r_1 \cdot 1 + r_2 \cdot d \tag{4.24}$$

for some $r_1, r_2 \in R$.

Corollary 10. *The topos $\mathbf{Set}^{(\mathbf{k\text{-}Alg}_{fp}^{op})^{op}}$ with $y(\mathbb{A})$ as its line object satisfies Axiom 1 and therefore is a model for synthetic differential geometry.*

There are generalizations of Axiom 1 that lead to a richer theory with more sophisticated differentials, which are obtained by factoring different ideals out of $k[X]$. For example, it is possible to define nilcube infinitesimals via $k[X]/(X^3)$; they can be used to calculate second derivatives. Furthermore, algebras like $k[X, Y]/(X^2, XY, Y^2)$ that lead to infinitesimals that can express partial differentials in two dimensional calculus, and so on. For more on those higher kinds of infinitesimals, refer to [Kock, 2006, I.3-6].

5 Grothendieck Topoi

This section is meant to give a quick definition of topoi, Grothendieck topologies and the way of obtaining the former from the latter. A very thorough but easy to read introduction is given in the first chapters of [Mac Lane and Moerdijk, 1992]. The rough idea stems from the following consideration:

- For a topological space X, the category $Sh(X)$ of sheaves of sets on it carries a nice structure which can give useful information about X and is also interesting to look at by itself.

- Given a category $\mathbf{C}$, the Yoneda embedding $y\colon \begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{Set}^{\mathbf{C}^{op}} \\ A & \longmapsto & \mathbf{C}(-, A) \end{array}$ is useful in many situations. It embeds $\mathbf{C}$ into $\mathbf{Set}^{\mathbf{C}^{op}}$, the category of contravariant set-valued functors, a.k.a.

presheaves, on $\boldsymbol{C}$. It has a logical structure that inherits many of the tools from $\boldsymbol{Set}$, in the sense that it has finite limits, is cartesian closed and has a subobject classifier. These properties make it what is defined as a topos. This allows some problems that pose themselves in $\boldsymbol{C}$ to be solved in $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$ making use of the constructions that a topos allows, and then to be carefully translated back.

- For a more interesting structure than the very set-like one of $\boldsymbol{Set}^{\boldsymbol{C}^{op}}$, we want to get the notion of sheaves on $\boldsymbol{C}$ by defining what covering families of an object in $\boldsymbol{C}$ are. The unique amalgamation criterion then tells us which functors actually are sheaves.

5.1 Sieves, sites and sheaves

Given an object A in a category $\boldsymbol{C}$, a sieve on A is a subfunctor of $\boldsymbol{C}(-, A)$. For a sieve S on B and a morphism $f : A \to B$, the limit of the diagram

(5.1)
$$\begin{array}{ccc} & & \boldsymbol{Set}(-, A) \\ & & \big\downarrow {\scriptstyle \circ f} \\ S & \overset{\iota_S}{\hookrightarrow} & \boldsymbol{Set}(-, B) \end{array}$$

is the pullback sieve f^*S on A.

A Grothendieck topology J on $\boldsymbol{C}$ assigns to each object A a set $J(A)$, the covering sieves of A, such that the following conditions are satisfied:

Identity $\boldsymbol{Set}(-, A) \in J(A)$.

Base change Pullbacks of covering sieves are covering.

Locality If $S \in J(A)$ and T is a sieve on A such that all pullbacks of T along morphisms in S are covering, then T itself is covering.

It is often more practical to regard a sieve as a set of actual morphisms with common codomain. This lets us express the above conditions for morphisms accoridng to [Mac Lane and Moerdijk, 1992, III.2, Definition 2]:

Identity $\mathbb{1} \in J(A)$.

Base change For $\{A_i \xrightarrow{g_i} A | i \in I\} \in J(A)$ and $f : B \to A$, the set of pullbacks $\{A_i \times_A B \to B | i \in I\}$ is in $J(B)$.

Locality Given a covering sieve $\{A_i \xrightarrow{g_i} A | i \in I\}$ on A and for each $i \in I$ a further one $\{A_{ij} \xrightarrow{g_{ij}} A | j \in J_i\}$ on A_i, then the sieve of composites $\{A_{ij} \xrightarrow{g_i \circ g_{ij}} A | i \in I,\ j \in J_i\}$ covers A.

A category $\mathbf{C}$ with a Grothendieck topology J on it is called the site $(\mathbf{C}, J)$.

A sheaf on a site $(\mathbf{C}, J)$ is a functor $F \in \mathbf{Set}^{\mathbf{C}^{op}}$ such that for each object A of $\mathbf{C}$ and each covering sieve $S \in J(A)$, precomposition with the inclusion $\circ \iota_S : Nat(\mathbf{C}(-, A), F) \to Nat(S, F)$ is a bijection

A site where for each object B of $\mathbf{C}$ the Hom-functor $\mathbf{C}(-, B)$ is a sheaf is called subcanonical. It is equivalent to demand for each covering sieve $S \in J(A)$ that $\circ \iota_S : (Nat(\mathbf{C}(-, A), \mathbf{C}(-, B)) =)\ \mathbf{C}(A, B) \to Nat(S, \mathbf{C}(-, B))$ is an isomorphism. If we interpret S as a collection $\{g_i \mid i \in I\}$ of arrows with codomain A, then $\circ \iota_S$ maps f to the collection $\{f \circ g_i \mid i \in I\}$. It is injective if $\forall i \in I : f_1 \circ g_i = f_2 \circ g_i$ implies $f_1 = f_2$. Thus, we can say that the topology $J(A)$ is subcanonical if each $\{g_1, g_2, g_3, \dots\} \in J(A)$ is collectively epimorphic.

5.2 The site of opens

Sheaves on a site are a generalization of sheaves on a topological space. To get familiar with the site formalism, we reproduces those by regarding the site of opens of a topological space X: Open subsets of X are the objects, inclusions are the morphisms. Therefore, a sieve on an open set U is a collection S of open subsets of U such that $V \in S$ implies that also all open subsets of V are in S. The set $S(V)$ contains one element if V belongs to the sieve and is empty otherwise.

Covering sieves on U are those whose union is all of U. This indeed defines a topology: The identity condition holds since the union of all open subsets of U obviously is U. It is clear that base change holds, as pullback of a collection of open sets along an inclusion $V \hookrightarrow U$ is the collection of the intersections of its members with V. Locality means that if $\bigcup_{V \in S} V = U$ and $\forall V \in S : (\bigcup_{W \in T} W \cap V) = V$, then $\bigcup_{W \in T} W = U$ - which is also clearly the case.

When is a contravariant functor on this category of open sets a sheaf? For simplicity, we write $|_W$ for $F(\iota_W^V) : F(V) \to F(W)$. Given a sieve S, elements of the set $Nat(S, F)$ of natural transformations from S to F are families of an $x_V \in F(V)$ for each $V \in S$, such that for $W \subseteq V$, it is $x_V|_W = x_W$. Thus, we call these natural transformations consistent families. The map $\iota_S : F(U) \to Nat(S, F)$ takes an $x \in F(U)$

and sends it to the family $(x|_V)_{V\in S}$. Surjectivity of that map means that for each consistent family $(x_V)_{V\in S}$ there is some $x \in F(U)$ with $x_V = x|V$. Injectivity means that there is at most one. Therefore, the notion of sheaf on the site of opens coincides with the topological one on X.

An important result is that for a site $(\mathbf{C}, J)$, the category $Sh(\mathbf{C}, J)$ of sheaves and natural transformations between them is a topos. This is expounded by the verification of all the relevant properties in [Mac Lane and Moerdijk, 1992, III.6/7].

6 C^∞-Rings as a Model

A branch of synthetic differential geometry that is particularly useful for doing differential geometry is the theory of smooth infinitesimal analysis. We follow the chapters I, II and VI of [Moerdijk and Reyes, 1991] to define the Zariski topos on the ground of smooth algebras.

6.1 The category C^∞-*Ring*

An $\mathbb{R}$-algebra A has the characterizing property that polynomial maps $P : \mathbb{R}^n \to \mathbb{R}^m, \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \mapsto \begin{pmatrix} \sum_{i\in\mathbb{N}^n} P_{1i} r_1^{i_1} r_2^{i_2} \cdots r_n^{i_n} \\ \sum_{i\in\mathbb{N}^n} P_{2i} r_1^{i_1} r_2^{i_2} \cdots r_n^{i_n} \\ \vdots \\ \sum_{i\in\mathbb{N}^n} P_{mi} r_1^{i_1} r_2^{i_2} \cdots r_n^{i_n} \end{pmatrix}$ can be lifted to maps from A^n to A^m, simply by letting the symbols r_j be elements of A.

The real numbers notoriously have more structure than just being a ring that allows the definition of polynomials. Continuous, differentiable and analytic functions come to mind. This is the idea behind the introduction of C^∞-rings: A C^∞-ring A is an $\mathbb{R}$-algebra A with the additional requirement that any smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ can be extended to a map (of the underlying sets) $A(f) : A^n \to A^m$ that is compatible with the algebra structure.

The definition of C^∞-rings can be expressed more precisely by the following description in terms of the category ***CartSp*** of cartesian spaces $\mathbb{R}^n$ and smooth maps between them: A C^∞-ring is a functor ***CartSp*** $\to$ ***Set*** which preserves finite products. Taking natural transformations as morphisms, we get a category $\boldsymbol{C^\infty}$***-Ring***. Given a functor A : ***CartSp*** → ***Set*** which is a C^∞-ring in the sense of this definition, we get an algebra $A(\mathbb{R})$ which is a C^∞-ring as in the preceding paragraph. We don't distinguish them in notation and write A for both. The category $\boldsymbol{C^\infty}$***-Ring*** has finite products, denoted by $\times$, and finite coproducts, denoted by $\otimes_\infty$.

For a class of examples of C^∞-rings, let X be a smooth manifold. Then $C^\infty(X,-) =: C^\infty(X)$ is the functor that assigns to $\mathbb{R}^n$ the set of smooth functions $X \to \mathbb{R}^n$. It is the covariant Hom-functor for X on manifolds restricted to ***CartSp***, which tells us that it indeed preserves limits since all covariant Hom-functors do [MacLane, 1998, Thm. 1 in ch. V.4]. That especially means $C^\infty(X, \mathbb{R}^n) = C^\infty(X, \mathbb{R})^n$. This is in accord with the fact from analysis that smoothness can be regarded componentwise.

The fundamental C^∞-rings for our theory are $C^\infty(\mathbb{R}^n)$. They are the free C^∞-rings in n variables:

Lemma 11.

$$\boldsymbol{C^\infty\text{-}Ring}(C^\infty(\mathbb{R}^n), A) = \{(a_1, \dots, a_n) \mid a_i \in A\}$$

as sets.

Proof: If we look at it from the functor perspective where $C^\infty(\mathbb{R}^n) = C^\infty(\mathbb{R}^n, -)$, Yoneda's Lemma says $\boldsymbol{C^\infty\text{-}Ring}(C^\infty(\mathbb{R}^n, -), A)) = A(\mathbb{R}^n)$. As A preserves products, this equals $A(\mathbb{R})^n = A^n$.

Viewed as algebras, the equality can be implemented by sending the projections $\pi_i : \mathbb{R}^n \to \mathbb{R}$ to a_i. This defines a smooth homomorphism $C^\infty(\mathbb{R}^n) \to A$ via $f \mapsto A(f)(a_1, \dots, a_n)$ in a unique and surjective way. □

In the case $n = 1$ this means that a smooth homomorphism from $C^\infty(\mathbb{R})$ to A is specified by the element of A that $\mathbb{1}_\mathbb{R}$, the identity on $\mathbb{R}$, gets mapped to. We can also conclude that a smooth homomorphism from $C^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^m)$ is determined by n smooth functions $\mathbb{R}^m \to \mathbb{R}$ that can be subsumed to just a single smooth function $\mathbb{R}^m \to \mathbb{R}^n$. The therewith apparent similarity of $C^\infty(\mathbb{R})$ with $k[X]$ in the k-algebraic model above will be extended in an analogous way, except that we will use a Grothendieck topology instead of simply accepting all presheaves. But first we make sure that we have everything we need for a line object.

A basic theorem from real analysis that will be useful is Hadamard's lemma.

Lemma 12 (Hadamard's Lemma)**.** *For any smooth function $f : \mathbb{R} \to \mathbb{R}$, there exists another smooth function g such that $f = f(0) + \mathbb{1}_\mathbb{R} \cdot g$.*

If we apply Hadamard's Lemma n times in a row, we get the n^{th} order Taylor polynomial plus some smooth function times $\mathbb{1}_R^n$ as remainder. For example, applying it twice yields $f = f(0) + \mathbb{1}_\mathbb{R} \cdot f'(0) + \mathbb{1}_\mathbb{R}^2 \cdot h$.

The maps that make $C^\infty(\mathbb{R})$ a coring are the same as those for $k[X]$ but replacing X by $\mathbb{1}_\mathbb{R}$. Note that 'taking the free object in n variables' is a functor which is left adjoint to the forgetful functor and as such preserves colimits. The coproduct of sets is their union. This means that the free C^∞-ring $C^\infty(\mathbb{R}^2)$ in two variables is the same as $C^\infty(\mathbb{R}) \otimes_\infty C^\infty(\mathbb{R})$, in parallel to $k[A, B]$. Written out, the coring maps are:

$$\overline{\alpha}\colon \begin{array}{ccc} C^\infty(\mathbb{R}) & \longrightarrow & C^\infty(\mathbb{R}^2) \\ \mathbb{1}_\mathbb{R} & \longmapsto & \pi_1 + \pi_2 \end{array}, \tag{6.1}$$

$$\overline{\mu}\colon \begin{array}{ccc} C^\infty(\mathbb{R}) & \longrightarrow & C^\infty(\mathbb{R}^2) \\ \mathbb{1}_\mathbb{R} & \longmapsto & \pi_1 \cdot \pi_2 \end{array}, \tag{6.2}$$

$$\overline{0}(\mathbb{1}_\mathbb{R}) = 0_\mathbb{R}, \ \ \overline{\nu}(\mathbb{1}_\mathbb{R}) = -\mathbb{1}_\mathbb{R}, \ \ \overline{1}(\mathbb{1}_\mathbb{R}) = 1_\mathbb{R}. \tag{6.3}$$

The projection maps π_i stand for the i^{th} variable of a function or the i^{th} entry of an input vector; for example, another way to write the coaddition would be $\overline{\alpha}(x \mapsto x) = \left(\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + y \right)$.

This constitutes a coring structure on $C^\infty(\mathbb{R})$ in the very same way the analogue definitions on $k[X]$ do.

An easy way of constructing new C^∞-rings is factoring out ring-theoretic ideals:

Lemma 13. *Given a C^∞-ring A and an ideal $I \leq A$, the quotient A/I is a C^∞-ring, too.*

Proof: For any smooth map $f : \mathbb{R} \to \mathbb{R}$ (let's stick to the one-dimensional case here, $f : \mathbb{R}^n \to \mathbb{R}^m$ works similar), we need to make sure that it preserves congruence modulo I. So for $a \in I$, we must show $A(f)(a) - A(f)(0) \in I$. We apply A to the equation $f(x) = f(0) + x \cdot g(x)$ from Hadamard's Lemma and get $A(f)(a) = A(f)(0) + a \cdot A(g)(a)$, where the latter summand is in the ideal I. □

A finitely generated C^∞-ring is one of the form $C^\infty(\mathbb{R}^n)/I$.

There is an easy way to describe functions ϕ between two finitely generated C^∞-rings $C^\infty(\mathbb{R}^n)/I$ and $C^\infty(\mathbb{R}^m)/J$. It is given by an equivalence class of maps $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ such that precomposition with φ maps I into J modulo each component of ϕ being in J. Another nice fact (cf. [Moerdijk and Reyes, 1991, after I.1.5]) is that factoring out ideals and tensoring are both colimits and therefore commute. In particular,

$$C^\infty(\mathbb{R}^n)/I \otimes_\infty C^\infty(\mathbb{R}^m)/J = C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I \circ \pi_1 + J \circ \pi_2)\ . \tag{6.4}$$

6.2 Loci

Above, we introduced affine schemes - which we defined as duals of k-algebras - as geometric counterparts of algebraic objects. Later, we restricted ourselves to finitely presented ones. Here, we do both at once. The object dual to a finitely generated C^∞-ring can be regarded as a geometric space and is called a locus. We write $\mathbb{L} = \boldsymbol{C^\infty\text{-}Ring}_{fg}^{op}$ for the category of loci. The dual of $C^\infty(\mathbb{R})$ is suggestively called R.

Inside $C^\infty(\mathbb{R})$, the functions f which allow factoring out a quadratic part, in the sense that there is some $g \in C^\infty(\mathbb{R})$ with $f(x) = x^2 \cdot g(x)$, form an ideal $(\mathbb{1}_{\mathbb{R}}^2)$. Factoring it out is a coequalizer.

$$C^\infty(\mathbb{R})/(\mathbb{1}_{\mathbb{R}}^2) \xleftarrow[\text{coeq}]{\mathbb{1}_{\mathbb{R}} \mapsfrom \mathbb{1}_{\mathbb{R}}} C^\infty(\mathbb{R}) \underset{\mathbb{1}_{\mathbb{R}}^2 \mapsfrom \mathbb{1}_{\mathbb{R}}}{\overset{\overline{0}}{\leftleftarrows}} C^\infty(\mathbb{R})\ . \tag{6.5}$$

Therefore its dual is the infinitesimal disk locus $D = \overline{C^\infty(\mathbb{R})/(\mathbb{1}_{\mathbb{R}}^2)}$.

Theorem 14. $R^D \xrightarrow{\sim} R^2$

Proof: We have to show that for any locus L, there is a natural bijection between $\mathbb{L}(L \times D, R)$ and $\mathbb{L}(L, R^2)$. The locus L is the dual of $C^\infty(\mathbb{R}^m)/J$ for some m and J. Passing to the opposite category, the bijection we have to establish must be between $\boldsymbol{C^\infty\text{-}Ring}(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}^m)/J \otimes_\infty C^\infty(\mathbb{R})/(\mathbb{1}_{\mathbb{R}}^2))$ and $\boldsymbol{C^\infty\text{-}Ring}(C^\infty(\mathbb{R}^2), C^\infty(\mathbb{R}^m)/J)$. The first set is $C^\infty(\mathbb{R}^m)/J \otimes_\infty C^\infty(\mathbb{R})/(\mathbb{1}_{\mathbb{R}}^2)$, since $C^\infty(\mathbb{R})$ being free in one variable means it just picks one element, and can be simplified to $C^\infty(\mathbb{R}^m \times \mathbb{R})/(J \circ \pi_1 + (\pi_2^2))$ as factoring and tensoring commute. The second is $(C^\infty(\mathbb{R}^m)/J)^2$, since a map from $C^\infty(\mathbb{R}^2)$ picks two elements. We define two maps

$$C^\infty(\mathbb{R}^m \times \mathbb{R})/(J \circ \pi_1 + (\pi_2^2)) \underset{s}{\overset{r}{\rightleftarrows}} (C^\infty(\mathbb{R}^m)/J)^2 \tag{6.6}$$

and show that they are mutually inverse.

The first one is

$$r : \psi \mapsto (\psi(-,0), \frac{\partial \psi}{\partial \pi_2}(-,0))$$

for $\psi = \psi(\pi_1, \pi_2) : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$. The derived function $\frac{\partial \psi}{\partial \pi_2}(-,0)$ assigns to each point z in $\mathbb{R}^n$ the derivative at zero of $\psi(z,-) : \mathbb{R} \to \mathbb{R}$. We have to make sure that this defines a function by checking that the image of some $j \circ \pi_1 + f \cdot \pi_2^2 \in J \circ \pi_1 + (\pi_2^2)$ under r is in J^2. It is indeed:

$$r(j \circ \pi_1 + f \cdot \pi_2^2) = (j + f(-,0) \cdot 0^2, 0 + \frac{\partial f}{\partial \pi_2}(-,0) \cdot 0^2 + f \cdot 2 \cdot 0) = (j,0) \in J^2 \tag{6.7}$$

The second map is

$$s : (f,g) \mapsto f \circ \pi_1 + \pi_2 \cdot g \circ \pi_1 \, . \tag{6.8}$$

We check that it defines a map by calculating that $(j_1, j_2) \in J^2$ is mapped into $J \circ \pi_1 + (\pi_2^2)$:

$$s(j_1, j_2) = j_1 \circ \pi_1 + \pi_2 \cdot j_2 \circ \pi_1 \in J \circ \pi_1 \tag{6.9}$$

Given a representative $\psi : \mathbb{R}^m \times \mathbb{R}$ of an element $\hat{\psi}$ of $C^\infty(\mathbb{R}^m \times \mathbb{R})/(J \circ \pi_1 + (\pi_2^2))$, we can twice apply Hadamard's Lemma to the second variable only and write

$$\psi = \psi(\pi_1, 0) + \pi_2 \cdot \frac{\partial \psi}{\partial \pi_2}(\pi_1, 0)) + \pi_2^2 \cdot h \tag{6.10}$$

. Only the first two summands are relevant, since π_2^2 is factored out. Therefore, its equivalence class modulo $J \circ \pi_1 + (\pi_2^2)$ is the same as that of

$$s(r(\psi)) = \psi(-,0) \circ \pi_1 + \pi_2 \cdot \frac{\partial \psi}{\partial \pi_2}(-,0)) \circ \pi_1 \, . \tag{6.11}$$

The other direction doesn't even touch representatives themselves:

$$r(s(f,g)) = (f + 0 \cdot g, 1 \cdot g + \pi_2 \cdot 0) = (f,g) \, , \tag{6.12}$$

since $\frac{\partial \pi_2}{\partial \pi_2} = 1$ and $g \circ \pi_1$ doesn't depend on π_2. □

6.3 The smooth Zariski Grothendieck topology

Given an element a of a finitely generated C^∞-ring A, we write $A[a^{-1}]$ for the localization of A at a. It still is finitely generated, since $C^\infty(\mathbb{R}^n)/I[a^{-1}] = C^\infty(\mathbb{R}^n \times \mathbb{R})/(I \circ \pi_1 + (\pi_2 \cdot a \circ \pi_1 - 1))$ It can also be expressed as pushout of localization by the identity in $C^\infty(\mathbb{R})$ along $\mathbb{1}_R \mapsto a$:

(6.13)
$$
\begin{array}{ccc}
C^\infty(\mathbb{R}) & \xrightarrow{\mathbb{1}_\mathbb{R} \mapsto a} & A \\
\downarrow & \text{pushout} & \downarrow \\
C^\infty(\mathbb{R})[\mathbb{1}_\mathbb{R}^{-1}] & \xrightarrow[\mathbb{1}_\mathbb{R}^{-1} \mapsto a^{-1}]{\mathbb{1}_\mathbb{R} \mapsto a} & A[a^{-1}]
\end{array}
.
$$

This lets us easily determine pushouts of localization by pushout pasting:

(6.14)
$$
\begin{array}{ccccc}
C^\infty(\mathbb{R}) & \xrightarrow{\mathbb{1}_\mathbb{R} \mapsto a} & A & \xrightarrow{f} & B \\
\downarrow & & \downarrow & & \downarrow \\
C^\infty(\mathbb{R})[\mathbb{1}_\mathbb{R}^{-1}] & \xrightarrow[\mathbb{1}_\mathbb{R}^{-1} \mapsto a^{-1}]{\mathbb{1}_\mathbb{R} \mapsto a} & A[a^{-1}] & \xrightarrow{a^{-1} \mapsto f(a)^{-1}} & B[f(a)^{-1}]
\end{array}
;
$$

we see that the pushout of the localization of A by a along f is simply B localized by $f(a_i)$.

We define the *smooth Zariski* Grothendieck topology on $\mathbb{L}$ as in [Moerdijk and Reyes, 1991, VI.1]: A covering sieve of a locus $\overline{A}$ is of the form $S = \{\overline{A \to A[a_1^{-1}]}, \dots, \overline{A \to A[a_n^{-1}]}\}$; it is given by finitely many elements $a_1, \dots, a_n \in A$ that generate A, i.e. $1 \in (a_1, \dots, a_n)$.

Lemma 15. *This indeed defines a Grothendieck-Topology.*

Proof: We check the conditions on morphisms:

Identity $\mathbb{1}_A = (A[1^{-1}] \to A)$.

Base change We start with a morphism $f : A \to B$ and $A \to A[a_1^{-1}] \ldots , A \to A[a_n^{-1}]$ such that $1 \in (a_1, \ldots , a_n)$. Pullback of loci corresponds to pushout of C^∞-rings. This means we have to make sure that $1 \in (f(a_1), \ldots , f(a_n))$. This is the case as

$$1 = \sum_{i=1}^{n} a_i \Rightarrow 1 = f\left(\sum_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} f(a_i) \, . \tag{6.15}$$

Locality Assume we are given $a_1, \ldots , a_n \in A$ with 1 being an A-linear combination of them and for each $i = 1, \ldots n$ elements $a_{i1}, \ldots a_{im_i}$ of $A[a_i^{-1}]$ with $1 = \sum_{j=1}^{m_i} P_{ij} a_{ij}$ for some $P_{ij} \in A[a_i^{-1}]$. Each $a_{ij} \in A[a_i^{-1}]$ can be written as $b_{ij} a_i^{-l_{ij}}$ with $b_{ij} \in A$. This allows us to rewrite $A[a_i^{-1}][a_{ij}^{-1}] = A[(b_{ij}a_i)^{-1}]$, since $a_i^{-1} = b_{ij}(b_{ij}a_i)^{-1}$ and $a_{ij}^{-1} = b_{ij}^{-1} a_i^{l_{ij}} = a_i^{l_{ij}+1}(b_{ij}a_i)^{-1}$ imply $A[a_i^{-1}][a_{ij}^{-1}] \subseteq A[(b_{ij}a_i)^{-1}]$, and $(b_{ij}a_i)^{-1} = a_{ij}^{-1} a_i^{-l_{ij}-1}$ implies the reverse inclusion. What we therefore have to show is that $1 \in (b_{ij}a_i | i \in I, j \in J_i)_A$.

For each i, we have $1 = \sum_{j=1}^{m_i} P_{ij} b_{ij} a_i^{-l_{ij}} = \sum_{j=1}^{m_i} Q_{ij} b_{ij} a_i$ with appropriate $Q_{ij} \in A[a_i^{-1}]$. We call k_i the highest exponent of a_i^{-1} in any of those Q_{ij} and multiply both sides with $a_i^{k_i}$ to get $a_i^{k_i} = \sum_{j=1}^{m_i} c_{ij}(b_{ij}a_i)$, this time with coefficients $c_{ij} \in A$. Thus $(a_i^{k_i} | i \in I)_A \subseteq (b_{ij}a_i | i \in I, j \in J_i)_A$. Let k be the largest of these k_i.

It is time to use the linear combination $1 = \sum_{i=1}^{n} c_i a_i$ from our first assumption: We take the $2k^{th}$ power of both sides and observe that all terms of the right hand side multinomial have a factor a_i^{k+x} for some i and are therefore in $(a_i^{k_i} | i \in I)_A$.

□

This topology is subcanonical [Moerdijk and Reyes, 1991, VI, Lemma 1.3]. The isomorphism defined in the proof of theorem 14 is induces an isomorphism of sheaves $\boldsymbol{C^\infty\text{-}Ring}(-, R^D) \xrightarrow{\sim} \boldsymbol{C^\infty\text{-}Ring}(-, R^2)$. We will conclude by realizing that theorem 14 can can be transported into the sheaf topos and that therefore we have a topos with a line object as in Axiom 1.

As [Mac Lane and Moerdijk, 1992, III.6] demonstrates, the calculation an exponential of sheafs can be performed by calculating the exponential of presheaves. Using the fact that the Yoneda embedding preserves exponentials once more, this means $\boldsymbol{C^\infty\text{-}Ring}(-, R^D) = \boldsymbol{C^\infty\text{-}Ring}(-, R)^{\boldsymbol{C^\infty\text{-}Ring}(-, D)}$ in the category of sheaves. In an analogous manner, with [Mac Lane and Moerdijk, 1992, III.4] it is clear that taking representable sheaves preserves limits. Therefore, $\boldsymbol{C^\infty\text{-}Ring}(-, R)$ is also a ring object, $\boldsymbol{C^\infty\text{-}Ring}(-, D)$ is its subobject of nilsquare elements as it was defined as an equalizer, and $\boldsymbol{C^\infty\text{-}Ring}(-, R^2) = \boldsymbol{C^\infty\text{-}Ring}(-, R)^2$. We conclude that the smooth Zariski topos with $\boldsymbol{C^\infty\text{-}Ring}(-, R)$

and its arithmetic operations satisfies Axiom 1 and hence is a model for synthetic differential geometry.

As discussed in [Moerdijk and Reyes, 1991, VI], this model has some desired properties that make it for some uses better suited than the presheaf topos on ***C^∞-Ring***, like the line object being a local ring and having internal arithmetics with infinitely large natural numbers.

References

[Bosch, 2014] Bosch, S. (2014). *Lectures on formal and rigid geometry*. Lecture notes in mathematics ; 2105. Springer, Cham.

[Bosch et al., 1984] Bosch, S., Güntzer, U., and Remmert, R. (1984). *Non-Archimedean analysis : a systematic approach to rigid analytic geometry*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen ; 261. Springer, Berlin.

[Kock, 2006] Kock, A. (2006). *Synthetic differential geometry*. London Mathematical Society lecture note series ; 333. Cambridge Univ. Press, Cambridge [u.a.], 2. ed. edition.

[Kock, 2010] Kock, A. (2010). *Synthetic geometry of manifolds*. Cambridge tracts in mathematics ; 180. Cambridge University Press, Cambridge [u.a.], 1. publ. edition. Includes bibliographical references and index.

[Mac Lane and Moerdijk, 1992] Mac Lane, S. and Moerdijk, I. (1992). *Sheaves in geometry and logic : a first introduction to topos theory*. Universitext. Springer, New York.

[MacLane, 1998] MacLane, S. (1998). *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York.

[Moerdijk and Reyes, 1991] Moerdijk, I. and Reyes, G. E. (1991). *Models for smooth infinitesimal analysis*. Springer, New York. Literaturverz. S. 395 - 399.

[Shulman, 2006] Shulman, M. (2006). Synthetic differential geometry.