The Tannaka Duality for Finite Groups with Arbitrary Fields

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Abstract

After defining the necessary concepts, we state, proof and discuss the Tannaka Duality for finite groups, which gives a way to recover a group from its tensor category of k-linear representations. Here k is an arbitrary field; this generalizes the statement from that of many publications, where certain properties of k, like being algebraically closed or of characteristic prime to the order of the group, are required. The Tannaka Duality is illustrated by the example of the groups D_8 and $\mathbb{H}_8$ which have the same representation category but different tensor structures on it. Finally we present some results that one can get for more general objects than finite groups.

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0.1 Notation

G will always denote a finite Group and k an arbitrary field. We denote by $k-mod$ the category of finite-dimensional k-vector-spaces.

1 Representations, the Group Algera and the Forgetful Functor F

Definition 1 (Representations)**.**

- *A (finite dimensional k-linear) representation (V, ϕ) (of G) is a k-vector-space V together with a group homomorphism ϕ from G to the group $Aut(V)$ of vector-space automorphisms of V.*
- *A morphism between two representations $(V, \phi), (W, \psi)$ is given by a linear map $f : V \longrightarrow W$ that is compatible with the action of G, which means that for any $g \in G$ the square*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi(g)"'] & W \arrow[d, "\psi(g)"] \\ V \arrow[r, "f"] & W \end{tikzcd} \tag{1}$$

  commutes.
- *One often writes $g \cdot v$ for $\phi(g)(v)$.*

With these definitions, the representations of G form a category; we call it $k - mod^G$. There are some constructions in this category that we will need.

Definition 2 (Some operations in the category of representations)**.**

- *The direct sum of two representations* (V, ϕ) *and* (W, ϕ) *is* $(V, \phi) \oplus (W, \phi) = (V \oplus W, \phi \oplus \psi)$ *.*
- *The tensor product of two representations* (V, ϕ) *and* (W, ψ) *is* $(V, \phi) \otimes (W, \psi) = (V \otimes W, \phi \otimes \psi)$ *where* $(\phi \otimes \psi)(g)(v \otimes w) = \phi(g)(v) \otimes \psi(g)(w)$*. We denote by* $(V, \phi)^{\otimes m}$ *the m-fold left tensor product of* (V, ϕ) *with itself.*
- *A subrepresentation of* (V, ϕ) *is a morphism* $\iota : (U, \nu) \longrightarrow (V, \phi)$ *that is an injective map of the vector spaces. One identifies it with its image* $(\iota(U), \iota \circ \nu)$*. This way, the subrepresentations of* (V, ϕ) *are those subspaces of* V *that are stable under the action of* G*.*
- *A representation is called irreducible if it has no subrepresentation not identifiable with itself or the zero-representation* $(\{0_V\}, g \mapsto id_{\{0_V\}})$
- *Let* (V, ϕ) *be a representation. We define the dual representation, by* (V^*, ϕ^*)*, where* $\phi^*(g)(f) \coloneqq f \circ \phi(g^{-1})$*.*
- *Let* $\underline{m} = (m_1, \ldots m_r), \underline{n} = (n_1, \ldots n_r) \in \mathbb{N}_0^r$ *and*

  $$(V, \phi)$$

  be a representation. Then denote

  $$T_{\underline{m},\underline{n}}(V, \phi) \coloneqq \bigoplus_{i=1}^{r} (V, \phi)^{\otimes m_i} \otimes (V^*, \phi^*)^{\otimes n_i}. \tag{2}$$

Definition 3 (Group Algebra)**.** *The group algebra* $k[G]$ *is the k-vector-space with basis* G *and multiplication based on the product in* G*.*

In $k[G]$ we have $1 = 1_k e_G$.

The Group algebra construction is free in the sense that for any group homomorphism f from G into the multiplicative group of an algebra A, there is exactly one algebra homomorphism $k[f] : k[G] \longrightarrow A$ making the following diagram commute:

$$\begin{tikzcd}
& k[G]^\times \arrow[d, "k[f]|_{k[G]^\times}^{A^\times}"] & k[G] \arrow[d, "k[f]"] \\
G \arrow[ur, hook] \arrow[r, "f"] & A^\times & A
\end{tikzcd} \tag{3}$$

For $g \in G$ the left multiplication with g is a k-linear automorphism ℓ_g of $k[G]$. This defines a representation ℓ.

It is easy to see that a representation is the same thing as a finite-dimensional $k[G]$-module and that the category of representations is isomorphic to the category of finite dimensional $k[G]$ modules $k[G] - mod$.

2 The Forgetful Functor $F : k - mod^G \longrightarrow k - mod$

Definition 4 (F). *We define* $F : k - mod^G \longrightarrow k - mod$, $(V, \phi) \mapsto V$

We take a closer look on the k-algebra $End(F)$ of natural transformations from F to itself and its subgroups $Aut(F)$ and $Aut^{\otimes}(F)$.

Per definitionem, an endomorphism $\tau \in End(F)$ is a family of linear maps $(\tau_{(V,\phi)} : F_{(V,\phi)} = V \longrightarrow V = F_{(V,\phi)})_{(V,\phi)\text{ representation}}$ such that for any morphism of representations $f : (V, \phi) \longrightarrow (W, \psi)$ the "naturality" square

$$\begin{tikzcd} V \arrow[r, "\tau_{(V,\phi)}"'] \arrow[d, "f"] & V \arrow[d, "f"] \\ W \arrow[r, "\tau_{(W,\psi)}"'] & W \end{tikzcd} \tag{4}$$

(where we write U for $F(U, \psi)$ and f for $F(f)$) commutes. $\tau_{(V,\phi)}$ is called the component of τ at (V, ϕ). The composition of two endomorphisms is the family consisting of the compositions of the components.

The multiplication with a fixed element of $k[G]$ on each representation as $k[G]$-module is an endomorphism of F (but not a tensor morphism); this gives $End(F)$ the structure of a $k[G]$-(left-)algebra.

If each $\tau_{(V,\phi)}$ is an automorphism of the vector space V, then the family of inverses $(\tau^{-1}_{(V,\phi)} : V \longrightarrow V)_{(V,\phi)\text{ rep.}} := \tau^{-1}$ is an endomorphism of F as well, and the compositions of the two yield id_F. So τ is an automorphism of F if and only if each component is an isomorphism of its vector space. The group of all automorphisms is called $Aut(F)$.

Considering the tensor structure on the category of representations, we can ask if an automorphism of F respects it. We call $\tau \in Aut(F)$ tensor automorphism and write $\tau \in Aut^{\otimes}(F)$ iff for any two representation (V, ϕ) and (W, ψ) the square

$$\begin{tikzcd} V \otimes W \arrow[r, "\tau_{(V,\phi)} \otimes \tau_{(W,\psi)}"'] \arrow[d, equal] & V \otimes W \arrow[d, equal] \\ V \otimes W \arrow[r, "\tau_{(V,\phi)\otimes(W,\psi)}"'] & V \otimes W \end{tikzcd} \tag{5}$$

commutes, i.e. $\tau_{(V,\phi)} \otimes \tau_{(W,\psi)} = \tau_{(V,\phi)\otimes(W,\psi)}$. For example, if $k = \mathbb{Q}$, the automorphism of F that is on any vector space V the multiplication with $a \in \mathbb{Q} - \{0, 1\}$ is not a tensor automorphism, as in general $av \otimes aw = a^2 v \otimes w \neq av \otimes w$. As compositions and inverses of tensor automorphisms are such as well, $Aut^{\otimes}(F)$ is a subgroup of $Aut(F)$.

3 The Tannaka-Duality for Finite Groups

For an element g of G, we define an endomorphism of F named $\gamma(g)$ as the family where each component $\gamma(g)_{(V,\phi)}$ is the automorphism $\phi(g)$. By definitions of the tensor product of representations and of vector space maps, we have

$$\begin{aligned} \forall V, W \forall v \in V, w \in W : \gamma(g)_{(V,\phi)\otimes(W,\psi)}(v \otimes w) &= (\phi(g) \otimes \psi(g))(v \otimes w) \\ = \phi(g)(v) \otimes \psi(g)(w) &= \gamma(g)_{(V,\phi)} \otimes \gamma(g)_{(W,\psi)}(v \otimes w)\,, \end{aligned}$$

hence $\gamma(g)_{(V,\phi)\otimes(W,\psi)} = \gamma(g)_{(V,\phi)} \otimes \gamma(g)_{(W,\psi)}$ and therefore $\gamma(g) \in Aut^{\otimes}(F)$. As the representation ℓ is faithful, γ is an embedding $G \hookrightarrow Aut^{\otimes}(F)$. The statement of the Tannaka-Duality for finite groups is that all tensor automorphisms are obtained this way.

Theorem 1 (Tannaka-Duality)**.**

$$\begin{aligned} \gamma : G &\longrightarrow Aut^{\otimes}(F) \qquad (6) \\ g &\longmapsto \gamma(g) \qquad (7) \end{aligned}$$

is a group isomorphism.

4 A proof of the Tannaka-Duality for Finite Groups

Proof of the Tannaka-Duality. By the freeness property of $k[G]$ we get a map $\kappa := k[\gamma|^{Aut(F)}] : k[G] \longrightarrow End(F)$. It is clear that κ coincides with the canonical mapping $- \cdot 1$ from $k[G]$ into the $k[G]$-algebra $End(F)$.

We define the map

$$\begin{aligned} \beta : End(F) &\longrightarrow k[G] \\ \tau &\longmapsto \tau_{k[G]}(1)\,, \end{aligned}$$

writing now and in the following simply $k[G]$ for the representation $(k[G], \ell)$; β is a morphism of algebras satisfying $\beta \circ \kappa = id_{k[G]}$. The latter implies that β is surjective. The situation combined in a diagram is the following:

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[d, hook, "\gamma"'] & k[G]^\times \arrow[r, phantom, "\subseteq"] \arrow[d, hook, shift left, "\kappa"] & k[G] \arrow[d, hook, shift left, "\kappa"] \\
Aut^\otimes(F) \arrow[r, phantom, "\subseteq"] \arrow[ur, "\beta^\otimes"] & Aut(F) \arrow[r, phantom, "\subseteq"] \arrow[u, shift left, "\beta"] & End(F) \arrow[u, shift left, "\beta"]
\end{tikzcd} \tag{8}$$

Our strategy is to show that $\beta^\otimes$ is actually an isomorphism between $Aut^\otimes(F)$ and G.

It remains to show that

s $\quad \beta(Aut^\otimes(F)) \subseteq G$. (Then $\beta^\otimes$ is a surjective map from $Aut^\otimes(F)$ to G because $\beta \circ \kappa = id_{k[G]}$ and therefore $\beta \circ \gamma = id_G$, which implies $\beta^\otimes(Aut^\otimes(F)) \supseteq G$.)

i $\quad \beta^\otimes$ is injective.

*Proof of **i**.* We proof the injectivity of the algebra homomorphism $\beta^\otimes$ by showing that even β is injective, which is equivalent to the fact that $\tau \in End(V)$, $\beta(\tau) = 0$ implies $\tau = 0$.

Let (V, ϕ) be any representation, i.e. a $k[G]$-module. We denote $dim_k(V) = d$. The module structure can be expressed as a map of k-vector-spaces

$$\begin{aligned} \pi : \quad k[G] \otimes_k V &\twoheadrightarrow V\,, \\ g \otimes v &\longmapsto \phi(g)(v) \end{aligned}$$

which is then by definition a map of $k[G]$-modules. As $k[G] \otimes_k V \cong k[G] \otimes_k k^d = k[G]^d$ as $k[G]$-modules, π defines an epimorphism from $k[G]^d$ to (V, ϕ).

Now let $\tau \in End(V)$ with $\beta(\tau) = 0$. Writing out the right multiplication representation, we get $\tau_{k[G]}(g) = \tau_{k[G]}(1 \cdot g) = \tau_{k[G]}(1) \cdot g = 0$, and by this $\tau_{k[G]} = 0$.

$k[G]^d$ is the sum of d copies of $k[G]$, on each of which the component of τ is zero. So by naturality of τ applied on the inclusions we get $\tau_{k[G]^d} = 0$.

By the naturality of τ we get the commutativity of the square

$$\begin{tikzcd}
k[G]^d \arrow[r, "0"] \arrow[d, "\pi"] & k[G]^d \arrow[d, "\pi"] \\
V \arrow[r, "\tau_{(V,\phi)}"] & V
\end{tikzcd}$$

which means $\tau_{(V,\phi)} \circ \pi = 0$, and the fact that π is an epimorphism yields $\tau_{(V,\phi)} = 0$. As (V, ϕ) was arbitrary, it follows that $\tau = 0$. $\square$

Before verifying **s**, we prove two general propositions about representations.

4.1 Some Statements About Affine Algebraic Groups

For the first proposition, we need some definitions from algebraic geometry, espacially the notion of an affine algebraic group. An affine algebraic variety V is defined by some ideal $(p_1, \dots, p_m) \subseteq [X_1, \dots, X_n]$ as the subset of $\mathbb{A}^n := k^n$ where every polynomial in I vanishes. A morphism of varieties is the restriction of a polynomial map between the $\mathbb{A}^i$s. To a variety V we associate the finitely generated algebra $k[V]$ of polynomial functions to k on V, called the coordinate ring of V. Furthermore, to a morphism $\phi : V \longrightarrow W$ we define $k[\phi] : k[W] \longrightarrow k[V]$ as precomposition with ϕ, i.e. $k[\phi](f) = f \circ \phi$. By this $k[-]$ is a contravariant functor that sends products to tensor products of algebras.

An affine algebraic group G is now a group object in the category of affine varieties, i.e. a variety G together with morphisms $\mu : G \times G \longrightarrow G$ and $\eta : \{0\} \longrightarrow G$ and $-^{-1} : G \longrightarrow G$ satisfying the common group axioms. Through $k[-]$, this produces algebra maps $k[\mu] : k[G] \longrightarrow k[G] \otimes k[G]$, $k[\eta] : k[G] \mapsto 0$ and $k[^{-1}] : k[G] \longrightarrow k[G]$.

An action of an affine algebraic group on a variety is a usual group action $\bullet : G \times V \longrightarrow V$ that is a morphism of varieties. It induces also an action of G on $k[V]$ via $g \star f := f \circ (g^{-1} \bullet \_)$.

For example any finite set can be seen a a variety and any finite group as a discrete affine algebraic group. Also linear groups of finite dimensional vector spaces are affine algebraic groups; their coordinate ring is $k[A_{11}, \dots, A_{dd}, Z]/(Z\,det(A)-1)$.

Lemma 5. *Let H be an affine algebraic group with an action $\bullet$ on a variety V and let $A < k[V]$ be a finite dimensional subspace of the coordinate ring. Then the space $H \star A = \{a \circ (h^{-1} \bullet \_) \mid a \in A, h \in H\}$ is finite dimensional.*

Note that $H \star A$ is invariant under the induced action $\star$ of H.

Proof of Lemma 5. As A is finite dimensional, it is the finite sum of one dimensional vector spaces A_j. Thus we can assume for the proof that A is one dimensional, say spanned by a single function a.

Under $k[-]$, the action gives an algebra map $k[\bullet] : k[V] \longrightarrow k[H] \otimes k[V]$, sending a to an element of the form

$$k[\bullet](a) = \sum_{i \in I} e_i \otimes f_i \quad \text{with } e_i \in k[H]\ ,\ f_i \in k[V] \text{ and } |I| < \infty. \tag{9}$$

Applying this to the definitions of the induced group action and the functor $k[-]$ on morphisms yields for any $v \in V$

$$h \star a = a(h^{-1} \bullet \_) = k[\bullet](a)(h^{-1}, \_) = \sum_{i \in I} e_i(h^{-1}) f_i(\_) = \sum_{i \in I} \underbrace{e_i(h^{-1})}_{\in k} f_i. \tag{10}$$

This means that for any $h \in H$ we have $h \star a \in \langle f_i \rangle_{i\in I}$. Hence $H \star a \subseteq \langle f_i \rangle_{i\in I}$ is finite dimensional. □

With help of this Lemma, we can prove the following proposition:

Proposition 6 (Chevalley's Theorem). *Let V be a finite dimensional k-vector-space and G an arbitrary closed algebraic subgroup of the affine algebraic group Gl_V. Then there is a representation (U, ψ) of Gl_V and a subspace $C \subseteq U$ such that $G = Stab(C) := \{f \in Gl_V \mid f|_C = id_C\}$.*

Proof of Proposition 6. We consider the variety Gl_V and the left action $\bullet$ of Gl_V on it, which is a not necessarily finite dimensional representation of Gl_V. The ideal $I := I(G) \lhd k[Gl_V]$ of functions vanishing on G is finitely generated, say by some elements a_i, as finite dimensional algebras are noetherian. Let A denote the vector space spanned by the a_i's. We apply Lemma 5 with $H \hookleftarrow Gl_V$ and $A \hookleftarrow A$ and see that $Gl_V \star A := E$ is finite dimensional and, as it is invariant, $(E, \star)$ is a representation of Gl_V.

We set $C := E \cap I$; it is invariant under the $\star$-action of G as E and I are (the latter because $g \in G, f \in I \Rightarrow g \star f(G) = f(g^{-1}G) = f(G) = 0$).

As G is closed, $V(I(G)) = G$, i.e. any common zero of I lies in G. If $h \in Gl_V$ stabilizes I then $h \in G$:

$$h \in Stab(I) :\Leftrightarrow \forall f \in I: \ 0 = (h \star f)(G) = f(h^{-1}G) \Rightarrow h^{-1}G = G \Rightarrow h \in G. \qquad (11)$$

If further an element l of Gl_V stabilizes C, since C contains the generators a_i of I, it follows

$$l \star I = l \star (Ck[Gl_V]) = (l \star C)(l \star k[Gl_V]) = Ck[Gl_V] = I, \qquad (12)$$

implying $l \in G$.

□

4.2 Affine Group Schemes

The goal of this subsection is to proove the following proposition:

Proposition 7. *Let W be a finite dimensional k-vector-space. Then any finite dimensional representation of Gl_W can be noted as a subrepresentation of a quotient of $T_{\underline{m},\underline{n}}W$ (for some $r \in \mathbb{N}_0, \underline{m} = (m_1, \dots m_r), \underline{n} = (n_1, \dots n_r) \in \mathbb{N}_0^r$) .*

The proof of this proposition can best be made with the notion of affine group schemes that we want to introduce in a very short manner, concentrating on the few facts that we need.

Definition 8 (Affine Group Schemes)**.** *The category of affine schemes over k is (equivalent to) the opposite of the category of k-algebras. (To be precise, an affine scheme (over k) is the spectrum, i.e. set of prime ideals, of a k-algebra A equipped with the Zariski topology whose closed sets are the $V(I)_{I \text{ ideal in } A}$, where $V(I)$ is the set of prime ideals containing I, and with the structure sheaf that is defined by sending the open sets $Spec(A) - V((f))$ to the localization of A at f.)*

An affine group scheme is now a group object in the category of affine schemes, that is a k-algebra A with three algebra maps

$$\begin{aligned} \Delta &: A \longrightarrow A \otimes_k A && (13) \\ \epsilon &: A \longrightarrow k && (14) \\ S &: A \longrightarrow A && (15) \end{aligned}$$

that fulfill the opposite properties of the group axioms associativity, identity and inversion.

From here on A denotes always an algebra with this opposite group structure that is anti-identified with the corresponding affine group scheme $Spec(A)$. An object with this structure is also called a Hopf-algebra. G denotes $Spec(A)$ with its group structure.

Definition 9. *A representation of A in a vector space V is an algebra map from the opposite of the affine group scheme Gl_V to A whose opposite is a group homomorphism. The representation form a category with morphisms linear maps that are compatible with the group action. Many definitions for affine group schemes are analogue to the ones in 2.*

For the Hopf-algebra A there is a dual concept to the one of a module:

Definition 10. *A comodule over A is a k-vector-space V with a k-linear map $\rho : V \longrightarrow V \otimes A$ that fullfills the opposite module axioms $(id \otimes \epsilon) \circ \rho = id_V$ and $(id_V \otimes \Delta) \circ \rho = (\rho \otimes id_A) \otimes \rho$. Morphisms between comodules are linear maps that satisfy the obvious compatibility conditions.*

Note that when A is a finitely generated k-algebra, i.e. $A = k[X_1 \dots X_n]/I$, then we have the situation of an affine algebraic group as defined above.

We will need the following two lemmata:

Lemma 11. *For a k-vector-space V one can find a one-to-one correspondence between the representations of G in V and the A-comodule structures on V.*

Proof. We choose a basis $(e_i)_{i \in I}$. If we have a anti-algebra map $\rho : G \mapsto Gl_V$, we identify $\phi(g) \in G$ with the corresponding matrix $(r(g)_{i,j})_{i,j} \in I$. The definition of a representation is satisfied if and only if ρ is a homomorphism of groups, i.e. that for any $g, h \in G$ and $i, j \in I$ we have:

$$r(gh)_{ij} = \sum_{k \in I} r(g)_{ik} r(h)_{kj}. \tag{16}$$

On the other hand, $\rho^{op} : V \longrightarrow V \otimes A$ as a k-linear map can be written as $\rho(e_j) = \sum_{i \in I} e_i \otimes r_{ij}$. With this, is it easy to see that ρ^{op} defines a comodule if and only if for any$i, j \in I$ we have:

$$\Delta(r_{ij}) = \sum_{k \in I} r_{ik} r_{kj} \tag{17}$$

Using the relation between Δ and its opposite, the multiplication in G, as well as the multiplication in A transferred to ist opposite, we get

$$\Delta(r_{ij})(gh) = r(gh)_{ij} \tag{18}$$

$$\sum_{k \in I} r_{ik} r_{kj}(gh) = \sum_{k \in I} r(g)_{ik} r(h)_{kj}. \tag{19}$$

Therefore the r_{ij} that define ρ determine that ρ is exactly then a representation when ρ^{op} defines a comodule structure on V. □

Lemma 12. *Every finite-dimensional representation is a subrepresentation of the n-fold sum of the regular representation, A^m, for some finite m.*

Proof. We regard the representation as comodule (W, ρ) and see the vector space W as isomorphic to k^m. The definition of (W, ρ) being a comodule shows that the diagram

$$\begin{tikzcd} W \arrow[r, "\rho"] \arrow[d, "\rho"] & A \otimes k^m = A^m \arrow[d, "\Delta \otimes \rho"] \\ A \otimes k^m = A^m \arrow[r, "id \otimes \rho"] & A \otimes A \otimes W \end{tikzcd} \tag{20}$$

commutes, which shows that $\rho : (W, \rho) \mapsto A^m$ is a morphism of comodules. As $W \cong k \otimes W$ in a natural way, we get the injectivity of ρ from the commutative triangle

$$\begin{tikzcd} W \arrow[r, "\rho"] & A \otimes k^m = A^m \arrow[d, "\epsilon \otimes id"] \\ & k \otimes W \arrow[ul, "\sim"] \end{tikzcd} \tag{21}$$

□

Now we can prove Proposition 7.

Proof of Proposition 7. Again we identify the representation with the corresponding comodule. The algebra opposite to Gl_V is $A = k[A_{11}, \dots, A_{dd}, Z]/(Z\, det(A) - 1)$. Z can be identified with $\frac{1}{det(A)}$, so we have a graduation by degree on A. Using Lemma 12, we just have to prove the statement for finite dimensional subcomodules of A^m; those are direct sums of their projections on the A's, therefore we can assume $m = 1$ and only have to deal with the subcomodules $k^n \simeq W \subseteq A$. As it is finite dimensional, W is for some $s, t \in (N)$ contained in $W' = \{Z^r p(A_{11}, \dots, A_{dd}) \mid deg(p) \leq s,\ r \leq t\} \subseteq A$ which is itself the direct sum of the $Z^r\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = c\}$ for $1 \leq d \leq s$. Let $(e_j)_{1\leq j\leq n}$ denote the standard basis of k^n. Expressed as comodule action, the regular representation is given by $e_j \longmapsto \sum_{i=1}^n e_j \otimes A_{ij}$. Here the map $e_j \mapsto A_{ij} \in A$ is a comodule morphism $k^n \longrightarrow A$. This means that also their sum $\bigoplus_{ij} k \cdot A_{ij} = \{p(A_{11}, \dots, A_{dd}) \mid deg(p) = 1\}$ is a subcomodule of the n^2-fold direct sum of the regular representation. The map

$$\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = 1\}^{\otimes c} \longrightarrow \{p(A_{11}, \dots, A_{dd}) \mid deg(p) = c\} \tag{22}$$

$$p_1(A_{11}, \dots, A_{dd}) \otimes \dots \otimes p_c(A_{11}, \dots, A_{dd}) \longmapsto p_1(A_{11}, \dots, A_{dd}) \cdot \dots \cdot p_c(A_{11}, \dots, A_{dd}) \tag{23}$$

is surjective; so $\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = c\}$ is a quotient cth tensor power of the subcomodule of the n^2-fold direct sum of the regular representation. In particular $\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = d\}$ is contained, which has the determinant $det(A)$ as an element. $det(A)$ corresponds to the determinant repesentation

$$det : Gl_V \longrightarrow Aut(k^1) = k \tag{24}$$

$$g \longmapsto det(g). \tag{25}$$

Because of the general fact $det(B^{-1}) = \frac{1}{det(B)}$, the definition of dual representations says that $\frac{1}{det}$, the corresponding to Z, is the dual representation of det. So Z is in the dual of $\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = d\}$ and Z^r in this duals rth tensor power. Tensoring it with $\{p(A_{11}, \dots, A_{dd}) \mid deg(p) = c\}$, we get W'. We have constructed W' and with that W from the regular representation only using subcomodules, direct sums, tensor powers and duals. □

4.3 Conclusion

Now we can proof the rest of the theorem:

Proof of **s**. We have $G \hookrightarrow k[G]^\times \hookrightarrow Gl_{k[G]}$ via left multiplication. By Proposition 6, there is a representation (U, ψ) of $Gl_{k[G]}$ and a subrepresentation $C \hookrightarrow (U, \psi)$ such that $G = Stab(C)$ Using Proposition 7 we regard (U, ψ) as subrepresentation of a quotient of $T_{\underline{m},\underline{n}}k[G]/N$. We will calculate in $T_{\underline{m},\underline{n}}k[G]$, as the fact of something being in the stabilizer of a subspace stays true if we go to the subqutioent. Now let $\tau \in Aut^{\otimes}(F)$ and $v \in T_{\underline{m},\underline{n}}k[G]$, written as $v = v_1 + \ldots + v_r$ with $v_i = x_{i_1} \otimes \ldots \otimes x_{i_{m_i}} \otimes f_{i_1} \otimes \ldots \otimes f_{i_{n_i}}$.

$$\begin{aligned}
\tau_{T_{\underline{m},\underline{n}}k[G]}(v) &= \tau_{\bigoplus_{i=1}^r k[G]^{\otimes m_i} \otimes (k[G]^*)^{\otimes n_i}}(v_1 + \ldots + v_r) && (26)\\
&= \sum_{i=1}^{r} \tau_{k[G]}^{\otimes m_i} \otimes \tau_{(k[G]^*)^{\otimes n_i}}(x_{i_1} \otimes \ldots \otimes x_{i_{m_i}} \otimes f_{i_1} \otimes \ldots \otimes f_{i_{n_i}}) && (27)\\
&\overset{Aut^{\otimes}}{=} \sum_{i=1}^{r} \tau_{k[G]}(x_{i_1}) \otimes \ldots \tau_{k[G]}(x_{i_{m_i}}) \otimes \tau_{k[G]^*}(f_{i_1}) \otimes \ldots \tau_{k[G]^*}(f_{i_{n_i}}) && (28)\\
&= \sum_{i=1}^{r} \beta(\tau) \cdot x_{i_1} \otimes \ldots \beta(\tau) \cdot x_{i_{m_i}} \otimes f_{i_1}(\beta(\tau) \cdot _) \otimes \ldots f_{i_{n_i}}(\beta(\tau) \cdot _) && (29)
\end{aligned}$$

which is the module-action of $\beta(\tau)$; this fact is passed to the transformation on the quotient $\tau_{W/N}$. As τ is a natural transformation, it stabilizes C:

$$\begin{tikzcd}
C \arrow[r, "\tau_C"] \arrow[d, hook] & C \arrow[d, hook] \\
T_{\underline{m},\underline{n}}k[G]/N \arrow[r, "\tau_{T_{\underline{m},\underline{n}}k[G]/N}"] & T_{\underline{m},\underline{n}}k[G]/N
\end{tikzcd} \tag{30}$$

We conclude that $\beta(\tau)$ is in the stabilizer of C, which is G. So we have showed **s** and completed the proof of the Tannaka Duality. □

□

5 Example: Distinguishing the Representation Tensor Categories of D_8 and $\mathbb{H}_8$

We want to demonstrate the possibility that two different groups have the same group algebra and character table but, as the Tannaka-Duality assures, differntly structured representation tensor categories. For this we take a look at two groups of order 8 and set for simplicity $k = \mathbb{C}$. Hereby we use some theory of irreducible representations to show some properties of the groups, but for the key statements in relation with the Tannaka-Duality it is not necessary to be familiar with it.

5.1 D_8

D_8 is the dihedral group with 8 elements, presented as $D_8 = \langle a, b | a^4 = 1 = b^4, b^{-1}ab = a^{-1}\rangle$. There are 5 conjugation classes of D_8:

$$\{1\}, \{r^2\}, \{r, r^3\}, \{s, sr^2\}, \{s, sr^2\}, \{sr, sr^3\} \tag{31}$$

There is a 2-dimensional representation of D_8, we call it $(\mathbb{C}^2, \rho)$ given in matrix form in the standard basis (e_1, e_2) by

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

$$s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{33}$$

This representation is irreducible. The commutator subgroup of D_8 shows to be the center with 2 elements. It follows that there are $\frac{8}{2} = 4$ irreducible representations of dimension 1. As the number of irreducible representations is the number of conjugation classes, those one 2-dimensional and 4 1-dimensional ones are the only irreducible representations of D_8.

We take a closer look at the tensor square representation $(\mathbb{C}^2, \rho) \otimes (\mathbb{C}^2, \rho)$. Its dimension is 4 and it is composed as sum of the 4 irreducible one-dimensional subrepresentations

$$V_1 := \langle e_1 \otimes e_1 + e_2 \otimes e_2 \rangle, \phi(r) = 1, \phi(s) = 1 \tag{34}$$

$$V_2 := \langle e_1 \otimes e_1 - e_2 \otimes e_2 \rangle, \phi(r) = -1, \phi(s) = -1 \tag{35}$$

$$V_3 := \langle e_1 \otimes e_2 + e_2 \otimes e_1 \rangle, \phi(r) = -1, \phi(s) = 1 \tag{36}$$

$$V_4 := \langle e_2 \otimes e_1 - e_1 \otimes e_2 \rangle, \phi(r) = 1, \phi(s) = -1. \tag{37}$$

Now let τ be a tensor automorphism of the forgetful functor F, and let

$$\tau_\rho = A = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in Gl_{\mathbb{C}^2} \tag{38}$$

be its component at $(\mathbb{C}^2, \phi)$. As it is a tensor automorphism, $\tau_{\rho \otimes \rho} = A \otimes A$.

Lemma 13. *Let (U, ϕ) be a trivial representation and τ a tensor endomorphism of F. Then $\tau_{(U,\phi)}$ is the identity.*

Proof. Choose a non-zero vector e from U. Then

$$\begin{aligned} I : U \otimes U &\longrightarrow U && (39)\\ ke \otimes le &\longmapsto kle && (40) \end{aligned}$$

is an isomorphism of representations. The linearity of $\tau_{(U,\phi)}$ means that there is a $\lambda \in k^{\times}$ such that $\tau_{(U,\phi)}(ke) = k\lambda e$ By naturality and tensor-functoriality of τ we get the following diagram:

$$\begin{tikzcd}
ke \otimes le \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \lambda ke \otimes \lambda le \arrow[dd, mapsto] \\
& U \otimes U \arrow[r, "\tau_{(U,\phi)} \otimes \tau_{(U,\phi)}"'] \arrow[d, "I"] & U \otimes U \arrow[d, "I"'] & \\
& U \arrow[r, "\tau_{(U,\phi)}"] & U & \lambda^2 kle \arrow[dl, equal] \\
kle \arrow[rr, mapsto] & & \lambda kle &
\end{tikzcd} \quad (41)$$

saying that λ must be 1. $\square$

We use this to have the fact that $A \otimes A$ acts trivial on V_1. This means that

$$\begin{aligned} e_1 \otimes e_1 + e_2 \otimes e_2 &= (A \otimes A)(e_1 \otimes e_1 + e_2 \otimes e_2) && (42)\\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_1) + (be_1 + de_2) \otimes (be_1 + e_2) && (43)\\ &= (a^2 + b^2)(e_1 \otimes e_1) + (ac + bd)(e_1 \otimes e_2 + e_2 \otimes e_1) + (c^2 + d^2)(e_2 \otimes e_2), && (44) \end{aligned}$$

so

$$\begin{aligned} a^2 + b^2 &= 1 && (45)\\ c^2 + d^2 &= 1 && (46)\\ ac + bd &= 0. && (47) \end{aligned}$$

Replacing the map I in Lemma 13 by a suitable map respecting the tensor product of the action of G one can see that also for non-trivial one-dimensional representations the component of τ must be the action of some element of G. Applied to V_2 this implies that $A \otimes A$ is there either the identity or the multiplication by -1. Like above, we get the following:

$$\begin{aligned} \pm(e_1 \otimes e_1 - e_2 \otimes e_2) &= (A \otimes A)(e_1 \otimes e_1 - e_2 \otimes e_2) && (48)\\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_1) - (be_1 + de_2) \otimes (be_1 + de_2) && (49)\\ &= (a^2 - b^2)(e_1 \otimes e_1) + (ac - bd)(e_1 \otimes e_2 + e_2 \otimes e_1) + (c^2 - d^2)(e_2 \otimes e_2), && (50) \end{aligned}$$

and by this the secound set of equations

$$\begin{aligned} a^2 - b^2 &= \pm 1 && (51)\\ d^2 - c^2 &= \pm 1 && (52)\\ ac + bd &= 0. && (53) \end{aligned}$$

The last equation of each set gives immediately $a = 0 \vee c = 0$ and $b = 0 \vee d = 0$. If $a = 0$ then $b \neq 0$, so $d = 0$, and doing this for all four one sees that $a = 0 = d$ or $b = 0 = c$. In the first case, it must be $b^2 = 1 = c^2$ and in the secound $b^2 = 1 = c^2$. So there are no more than 8 possibilities for the entries of the matrix A. But the group action of D_8 gives already 8 different matrices of tensor automorphisms. This demonstrates that the tensor automorphisms of τ are the same as the group D_8.

5.2 $\mathbb{H}_8$

$\mathbb{H}_8$ is the quaternion group, presented as $\langle j, k | j^4 = 1 = k^4, j^2 = k^2, kj = jk \rangle$(, where as for the quaternions we can see jk as i and $j^2 = k^2 = i^2$ as -1). With the same arguments as for D_8 we find that there are four 1-dimensional and one 2-dimensional representations, the latter, called ψ, is given by:

$$\begin{aligned} j &\mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} && (54)\\ k &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. && (55) \end{aligned}$$

This time, the tensor square $(\mathbb{C}^2, \phi) \otimes (\mathbb{C}^2, \phi)$ decomposes into the same vector spaces as for D_8 with permutated actions:

$$\begin{aligned} W_1 &:= \langle e_1 \otimes e_1 + e_2 \otimes e_2 \rangle, \phi(r) = -1, \phi(s) = 1 && (56)\\ W_2 &:= \langle e_1 \otimes e_1 - e_2 \otimes e_2 \rangle, \phi(r) = -1, \phi(s) = -1 && (57)\\ W_3 &:= \langle e_1 \otimes e_2 + e_2 \otimes e_1 \rangle, \phi(r) = 1, \phi(s) = -1 && (58)\\ W_4 &:= \langle e_2 \otimes e_1 - e_1 \otimes e_2 \rangle, \phi(r) = 1, \phi(s) = 1. && (59) \end{aligned}$$

We take again the matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl_{\mathbb{C}^2} \quad (60)$$

of the componenent at ψ of some tensor automorphism of the forgetful functor F for $\mathbb{H}_8$. We do the same calculation as above for the trivial representation, which is here W_4:

$$\begin{aligned} e_1 \otimes e_2 - e_2 \otimes e_1 &= (B \otimes B)(e_1 \otimes e_2 + e_2 \otimes e_1) && (61)\\ &= (ae_1 + ce_2) \otimes (be_1 + de_2) - (be_1 + de_2) \otimes (ae_1 + ce_2) && (62)\\ &= (ad - bc)(e_1 \otimes e_2 - e_2 \otimes e_1), && (63) \end{aligned}$$

gives

$$ad - bc = 1 \qquad (64)$$

For W_1 and W_2 we get the same equations as above for V_1 and V_2, with the difference that in the first equation for W_1 ± 1 and not only $+1$ are allowed. In total we have the system of equations

$$\begin{aligned} ad - bc &= 1 && (65)\\ a^2 + b^2 &= \pm 1 && (66)\\ c^2 + d^2 &= \pm 1 && (67)\\ a^2 - b^2 &= \pm' 1 && (68)\\ c^2 - d^2 &= \pm' 1 && (69)\\ ac - bd &= 0. && (70) \end{aligned}$$

There are again only 8 solutions for this system of equations which all must be already covered by the tensor automorphisms coming from the group $\mathbb{H}_8$ itself.

So we have two different results for a property of the tensor category of representations with its forgetful functor despite the ordinary categories $\mathbb{C}[G]-mod$ are the same.

6 Generalizations

6.1 Tannaka-Duality for Compact Topological Groups

In the same way as we defined an affine algebraic group as a group object in the category of varieties over a field, one gets the notion of a compact topological group as a group object in the category of continous maps between compact topological spaces. So a topological group is a group whose underlying "set" is a topological space such that the group multiplication and inversion are continous. A finite dimensional $\mathbb{C}$-linear representation is a representation as defined for finite groups such that the map $\bullet G \otimes V \longrightarrow V$ is continous for the standard $\mathbb{C}$-vector-space topology.

With the same definitions as above and the definition of a topology on natural endomorphisms of the forgetful functor F as the coarsest topology to make each taking of a component continous with respect to the natural topology of matrices, we get the same statement of Tannaka Duality as for finite groups and arbitrary fields:

Theorem 2. *$Aut^{\otimes}(F)$ is compact and*

$$\gamma : k - mod^{G} \longrightarrow Aut^{\otimes}(F) \tag{71}$$

is an isomorphism of compact groups.

However, some steps in the proof have to be done in a different way. The most important thing is that Proposition 7 doesn't hold anymore if V, which will later be used as $k[G]$, is not finite-dimensional any more.

6.2 Tannakian Categories

The Tannaka Duality can be regarded in a much wider context describing the interplay between Tannakian categories and affine group schemes. Let still denote k a field and $G = Spec(A)$ an affine group scheme.

The forgetful functor F is again defined as assigning a representation of G its vector space.

Theorem 3 (Tannaka Duality for Affine Group Schemes)**.** *The natural map $\gamma : G \longrightarrow Aut^{\otimes F}$ is an isomorphism.*

We define a type of categories with a special tensor structure on it that generalizes the repesentation category of a group.

Definition 14 (Definition of Neutral Tannakian Categories)**.**

- *A tensor category C is a category with a functor $C \times C \longrightarrow C$ and a unit object I and isomorphisms satisfying axioms for associativity, identity and coherence.*
- *Such a tensor category is rigid, iff to a pair of objects belongs a "Hom"-object of C in a natural and with $\otimes$ compatible way.*
- *It is abelian rigid tensor category when it is abelian and direct sum and $\otimes$ are distributive.*
- *If $End(I) = k$, we call it abelian rigid tensor category over k.*

- *A neutral tannakian category is an abelian rigid tensor category over k with an exact faithful k-linear tensor functor F into the category of k-vector-spaces, called fiber functor.*

Examples for neutral tannakian categories are the category of k-vector-spaces, the category of finite dimensional k-vector-spaces - for both F is the identity/inclusion - and the category of representations of any affine group scheme.

Theorem 4. *Let C be a neutral tannakian category with $End(I) = k$ and let $F : C \longrightarrow k - mod$ be an exact faithful k-linear tensor functor. Then $Aut^{\otimes F}$ is representable by some affine group scheme G and C is canonically equivalent to the category $rep_k(G)$ of representations of G in k-vector-spaces.*

A proof of the last two theorems is given in "James Milne & Pierre Deligne: Tannakian Categories".

With the latter theorem one can find many correspondences between certain properties of the neutral tannakian category $rep_k(G)$ and the group scheme. For example one can show that G being a finite group is equivalent to $rep_k(G)$ having an object $k\langle G\rangle$ such that every object in $rep_k(G)$ is a subquotient of finitely many copies of $k\langle G\rangle$. This object is, as seen in the theory of finite dimensional representations of finite groups, the group algebra $k[G]$. A similar statement is that G is an algebraic group if and only if $rep_k(G)$ has an object X such that every object in $rep_k(G)$ is a subquotient of $T_{\underline{m},\underline{n}}X$. We have seen the one implication explicitely for the case $G = Gl_V$ with $X = V$.

7 References

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The main reference was the article by Andrew Dudzik, from where the example with D_8 and $\mathbb{H}_8$ comes.

- Andrew Dudzik: Tannaka-Krein Duality for Finite Groups, 2011

- Armand Borel: Linear Algebraic Groups, Springer 1969

- James Milne: Basic Theory of affine Group Schemes, 2012

- Yves André: Une Introduction aux Motifs, SMF 2004

- Anna Cadoret: Notes to the course "Groupes et Représentations X2012/13", 2012

- Jean-Pierre Serre: Linear Representations of Finite Groups, Springer 1977
- Alain Robert: Introduction to the Representation Theory of Compact and Locally Compact Groups, Cambridge UP 1983
- James Milne & Pierre Deligne: Tannakian Categories. In: Lecture Notes. Hodge Cycles, Motives and Shimura Varieties, Springer 1982
- André Joyal & Ross Street: An Introduction to Tannaka Duality and Quantum Groups, In:Lecture Notes. Category Theory, Proceedings, Como 1990, Springer 1991
- Neantro Rivano: Lecture Notes. Catégories Tannakiennes, Springer 1972