M2 Analyse, Arithmétique et Géométrie

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The Eilenberg-MacLane Theorem for Simplicial Sheaves

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Abstract. For a simplicial set X and an abelian group M there is the classical result by Eilenberg and MacLane that the n-th simplicial homology group $H^n(X, M)$ of X with coefficients in M is isomorphic to the group $[X, K(M, n)]$ of morphisms in the homotopy category of simplicial sets from X to the Eilenberg-MacLane space $K(M, n)$. In this paper, that result is expanded to the case where X is a simplicial sheaf and M is a sheaf of abelian groups over some site.

Contents

1. Introduction

This section's purpose lies in introducing the things that will appear in the theorem and its proof as well as stating some basic lemmas for them.

1.1. **Simplicial Sheaves.** Let (C, θ) be a site, that is a category C with a Grothendieck topology θ. This means (see for example chapter III of [MacLane and Moerdijk, 1992]) that to every object U in C is assigned a set $\theta(U)$ of subfunctors $R \subseteq C(-, U)$ such that

(1) $R \in \theta(U), f : V \to U \Rightarrow f^*R \in \theta(V)$
(2) $R \in \theta(U), S \subseteq C(-, U), (f : V \to U \in R(V) \Rightarrow f^*S \in \theta(V)) \Rightarrow S \in \theta(U)$
(3) $C(-, U) \in \theta(U)$.

A simplicial presheaf is a contravariant functor from C into the category $Set^{\Delta^{op}}$ of simplicial sets. With natural transformations as morphisms, the simplicial presheaves form a category $SPre(C) = (Set^{\Delta^{op}})^{C^{op}}$. For a presheaf $X \in SPre(C)$ and $R \subseteq C(-, U)$ the limit $\lim_{f:V \to U \in R} X(V) =: X(U)_R$ exists. The maps $X(f) : X(U) \to X(V)$ provide a canonical map $\tau_R : X(U) \to X(U)_R$.

Now X is a simplicial sheaf if for every object $U \in C$ and $R \in \theta(U)$ the map τ_R is an isomorphism (this definition is from [Jardine, 2007, p.37]). Note that an equivalent way to define simplicial sheaves would be as simplicial objects in the category of sheaves. The simplicial sheaves form a full subcategory $SSh(C)$ of $SPre(C)$ and there is a sheafification functor that turns a presheaf into a sheaf and is left adjoint to the inclusion functor $SSh(C) \hookrightarrow SPre(C)$.

The usual constructions that are possible for simplicial sets, like small limits and colimits, can be transferred to the category $SSh(C)$ by executing them on the stalks and if necessary using sheafification thereafter.

For a simplicial set A there is the so called constant presheaf that sends every object in C to A. The sheafification of this is the constant sheaf associated to A and often simply denoted A.

If we replace the category Set of sets with that of abelian groups Ab, we get simplicial sheaves of abelian groups, denoting their category $SSh_{Ab}(C)$. Here everything is the same as for sets with morphisms also having to be group homomorphisms on the sections.

We will also need sheaves of chain complexes of abelian groups, that among other ways can be defined similar to sheaves of abelian groups (those e.g. seen as simplicial sheaves of abelian groups concentrated in simplicial degree 0), by replacing the target category to that of chain complexes. Until mentioned otherwise, all chain complexes will be bounded below at zero, i.e. they dont have nontrivial entries at degree < 0.

1.2. **Model Structures.** A model structure (after [Quillen, 1967, p.1]) on a category consists of three classes of maps: weak equivalences W, cofibrations C and fibrations F, that satisfy the following axioms.

M0: The underlying category is complete and cocomplete.

M1f: Fibrations have the right lifting property with respect to all trivial cofibrations (i.e. cofibrations that are weak equivalences):

In any square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\wr}\big\downarrow i & & \big\downarrow p \\ B & \longrightarrow & Y \end{array} \tag{1.1}$$

where $p \in F$ and $i \in W \cap C$ and the vertical arrows are arbitrary morphisms, there is a morphism $k : B \to X$ such that the diagram

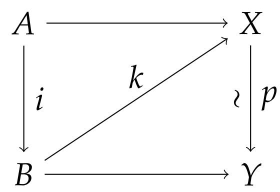
$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle\wr}\big\downarrow i & \overset{k}{\nearrow} & \big\downarrow p \\ B & \longrightarrow & Y \end{array} \tag{1.2}$$

commutes.

M1c: Cofibrations have the left lifting property with respect to all trivial fibrations (i.e. fibrations that are weak equivalences): For

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow i \in C & & {\scriptstyle\wr}\big\downarrow p \in W \cap F \\ B & \longrightarrow & Y \end{array} \quad , \tag{1.3}$$

the map k in



(1.4)

exists.

Note that the liftings k are by no means unique.

M2c: Any map can be factored as a composition of a trivial cofibration followed by a fibration.

M2f: Any map can be factored as a composition of a cofibration followed by a trivial fibration.

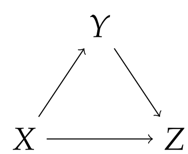
M3f: F is stable under composition and pullback and contains the class of all isomorphisms.

M3c: C is stable under composition and pushout and contains the class of all isomorphisms.

M4f: Pullbacks of trivial fibrations are trivial fibrations.

M4c: Pushouts of trivial fibrations are trivial cofibrations.

M5: If in a diagram



(1.5)

two of the three arrows are weak equivalences, then also the third. Furthermore, W contains the class of all isomorphisms.

A model category is a category together with a model structure on it.

An object A is called cofibrant if the morphism from the initial object into A is a cofibration and it is called fibrant if the morphism from A into the final object is a fibration.

The category of simplicial sets with weak homotopy equivalences as weak equivalences, injective maps as cofibrations and Kan fibrations as fibrations is a model category (see [Quillen, 1967, p.1.3]).

It is a known theorem that given two of the three classes W, C, F, the thitd one is uniquely determined.

For a model category, we define the homotopy category which is constructed by a process called localization that essentially factors out weak equicalences, making them isomorphisms (for the detailed construction refer to [Quillen, 1967, p.1.12] .

1.2.1. *For Simplicial Sheaves.* On the category of simplicial sheaves, there are several possible model structures, but the usual ones only differ on the definitions of fibrations and cofibrations. Hence the validity of statements made about the homotopy category, which is defined by making weak equivalences isomorphisms, are independent of this choice. We take the definition also used in [Morel and Voevodsky, 1999, p.48].

- **W:** The class of weak equivalences consists of sheaf morphisms that are weak equivalences of simplicial sets on all stalks.

For the proof will use the following notion, that is presented and proved to be a model structure in [Morel and Voevodsky, 1999, p.48], of fibrations and cofibrations:

- **C:** The cofibrations are the monomorphisms. Note that being a monomorphism of sheaves is eqivalent to being mono on the stalks.
- **F:** A morphism is a fibration if it has the right lifting property with respect to all trivial cofibrations (i.e. morphisms in $W \cap C$).

For simplicial sheaves of abelian groups the definitions are the same, only depending on the underlying simplicial sheaf of sets.

1.2.2. *For Sheaves of Chain Complexes.* For the category of sheaves of chain complexes of abelian groups on C, we use a similarly defined model structure (in details in [Hovey, 1999]) , which later allows us to switch between the two concepts via Dold-Kan correspondence.

- **W:** The weak equivalences are those morphisms that are quasi-isomorphisms on all stalks. A quasi-isomorphism is a chain complex map that induces an isomorphism in chain complex homology.
- **C:** Again, the cofibrations are the monomorphisms. A morphism is mono if it is a mono of chain complexes on all stalks. The latter is the case if a map is mono in each degree.
- **F:** Consequently, the fibrations are those morphisms that have the right lifting property with respect to all trivial cofibrations.

As morphisms from the initial object are always mono, all objects in those model categories defined above are cofibrant.

1.3. **Free Simplicial Sheaf of Abelian Groups.** Given a simplicial set A we can form the simplicial abelian group $\mathbb{Z}[A]$ with vertex groups $\mathbb{Z}[A]_n = \mathbb{Z}[A_n]$, i.e. the free abelian groups over the vertex sets, and simplicial structure maps induced by those on the sets. If we regard a simplicial set as a functor $A : \Delta^{op} \to Set$, this process is just postcomposition with the free abelian group functor $\mathbb{Z} : Set \to Ab$.

Note that the free simplicial abelian group functor $\mathbb{Z}[-] : SSet \to SAb$ is compatible with the model structures.

To a simplicial sheaf of sets X we assign the free simplicial sheaf of abelian groups $\mathbb{Z}[X]$ as the sheafification of the presheaf of simplicial abelian groups $U \mapsto \mathbb{Z}[X(U)]$.

This free simplicial sheaf of abelian groups functor does preserve the model structure as well, see [nla, c] and [Morel and Voevodsky, 1999, p.58].

Like the basic free abelian group functor, $\mathbb{Z}[-]$ is left adjoint to the forgetful funtor that assigns a group its underlying set.

1.4. **Dold-Kan Correspondence.** The classical Dold-Kan correspondence is a pair of functors

$$Ab^{\Delta^{op}} \underset{\Gamma}{\overset{N}{\rightleftarrows}} ChAb_+ \tag{1.6}$$

that form an equivalence between the categories of simplicial abelian groups and chain complexes of abelian groups.

Here the functor N is constructed in the following way: The Moore-Complex associated to a simplicial abelian group A is a chain complex with the group of n-simplices A_n at degree n and the map

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i : A_n \to A_{n-1} \tag{1.7}$$

as n-th boundary map. Calculating with simplicial identities yields $\partial_{n-1}\partial_n = 0$, therefore it is a chain complex. Now we regard $DA_n = \bigoplus_{i=0}^{n-1} Im(s_i) \leq A_n$, the subgroup of degenerate simplices; we have, $\partial_n(DA_n) \leq DA_{n-1}$. Thus ∂_n projects to a map on the quotients

$$\partial_n : A_n/DA_n \to A_{n-1}/DA_{n-1}. \tag{1.8}$$

This resulting chain complex of the quotients is NA and the construction assures that N is a functor. For a precise proof of that fact and the defintion of the map Γ refer to [Goerss and Jardine, 1999, chapter III.2].

Furthermore, the equivalence respects the model structures as weak equivaleces and monomorphisms are preserved in both directions; especially, simplicial homotopies correspond to chain homotopies.

After [Morel and Voevodsky, 1999, p.56], this construction is also possible for sheaves by applying N respectively Γ pointwise and then using sheafification. Thus we have an equivalence

$$SSh_{Ab}(C) \underset{\Gamma}{\overset{N}{\rightleftarrows}} Sh_{ChAb_+}(C) \tag{1.9}$$

that also preserves the model structures.

1.5. **Eilenberg-MacLane Objects.** For an abelian group M, let $M[n]$ be the chain complex

$$0 \leftarrow \cdots \leftarrow 0 \leftarrow 0 \leftarrow M \leftarrow 0 \leftarrow 0 \leftarrow \cdots \tag{1.10}$$

where all terms are zero except for the n-th which is M.

Now we apply the functor Γ from the Dold-Kan correspondence to get the simplicial set $K(M, n) = \Gamma(M[n])$. This simplicial set is a classical Eilenberg-MacLane space, characterized by the property that its siplicial homology groups $\pi_i(K(M, n))$ are trivial for $i \neq n$ and the group M for $i = n$.

The same procedure can (again, see [Morel and Voevodsky, 1999, p.56]) be applied for a sheaf of abelian groups $M \in Sh_{Ab}(C)$. Here the sheaf complex $M[n] \in Sh_{ChAb_+}(C)$ also is zero in all degrees but the n-th where it is equal to M; that is with the sheaf M if regarded as chain complex of sheaves respectively with the section of M in each section if seen as sheaf of chain complexes. Now we use the Dold-Kan correspondence for sheaves, getting the simplicial sheaf of abelian groups $K(M, n) = \Gamma(M[n])$.

1.6. **Injective Objects.** In general, an object I is called "injective" if it fullfills the universal mapping property that for any morphism $f : A \to I$ and any injection $g : A \hookrightarrow B$, there is a morphism $h : B \to I$ with $hg = f$. Speaking with diagrams, that means that for any diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & I \\ {\scriptstyle g}\big\downarrow & & \\ B & & \end{array} \tag{1.11}$$

a diagonal arrow h such that

$$\begin{array}{ccc} A & \xrightarrow{f} & I \\ {\scriptstyle g}\big\downarrow & \nearrow_{h} & \\ B & & \end{array} \tag{1.12}$$

commutes exists.

We will often use the property that there are "enough injectives". That means that for any object A, there is an injective resolution, that is a long exact sequence

$$0 \to A \to I_1 \to I_0 \to I_2 \to \cdots \tag{1.13}$$

where the I_i are injective objects.

From the facts that the categories Ab of abelian groups and the category of sheaves $Sh(C)$ have enough injectives, it is possible to derive the existence of enough injectives for every category that we will deal with. Those facts are shown in [nla, b].

1.7. **Some Lemmas.** The following two general lemmas will later come in handy for the main theorem's proof.

Lemma 1. *If*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{1.14}$$

is a fibre sequence, then

$$[X, A] \xrightarrow{\alpha^*} [X, B] \xrightarrow{\beta^*} [X, C] \tag{1.15}$$

is exact at $[X, B]$.

For the proof see [nla, a]; it mainly uses the the *Hom*-functor preserves pullbacks and that in this case the violation of right exactness vanishes in homotopy.

Lemma 2. *In an abelian category, the pushout of a monomorphism is also a monomorphism.*

Proof. We want to prove that in a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{p'} & E \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ A & \xrightarrow{p} & M \end{array} \tag{1.16}$$

where i is a monomorphism, the same is true for i'.

One calculates that there is a short exact sequence

$$0 \longrightarrow A \xrightarrow{i-p} B \oplus M \xrightarrow{p'+i'} E \longrightarrow 0 \tag{1.17}$$

and a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \xrightarrow{0} & M & \xrightarrow{Id} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0+Id} & & \downarrow{\scriptstyle i'} & & \\ 0 & \longrightarrow & A & \xrightarrow[i-p]{} & B \oplus M & \xrightarrow[p'+i']{} & E & \longrightarrow & 0 \end{array} \tag{1.18}$$

We add the kernels and cokernels of the vertical maps to the diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & 0 & \longrightarrow & ker(i') & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \xrightarrow{0} & M & \xrightarrow{Id} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0+Id} & & \downarrow{\scriptstyle i'} & & \\ 0 & \longrightarrow & A & \xrightarrow[i-p]{} & B \oplus M & \xrightarrow[p'+i']{} & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A & \xrightarrow{i} & B & \longrightarrow & coker(i') & & \end{array} \tag{1.19}$$

Now we use the snake lemma, which gives us an exact sequence

$$0 \to 0 \to ker(i') \xrightarrow{\partial} A \xrightarrow{i} B \to coker(i'). \tag{1.20}$$

Exactness at $ker(i')$ of course gives that ∂ is injective while exactness at A yields $im(\partial) = ker(i) = 0$ as i is a monomorphism. Thus $ker(i')$ is injected into 0, hence trivial.

□

1.8. **Prerequisites Used for the Main Theorem.** For the purpose of making the proof more straightforward, we make the following restrictions. Note that the theorem and most lemmas stay true in the general case, but would need some more complicated techniques and results to be proven beforehand.

The site C should have enough points. That means that certain properties of sheaves can be tested stalkwise. If for a morphism $f : X \to Y$ of sheaves on C is a mono/epi/iso on all stalks, it is a mono/epi/iso itself.

Also, we assume that there are no set-theoretic problems of any kind, so that for example the morphism classes in the homotopy category are sets. For this it might be needed that the category C is small. We also use some form of axiom of choice to make sure that Ab has enough injectives.

The object X should be a sheaf without simplicial structure. If we still regard X as a simplicial sheaf, it only has 0-vertices.

2. $[X, K(M, n)]$, $H^n(X, M)$ and a Map between them

In this section we construct a natural map $[X, K(M, n)] \to H^n(X, M)$ which will later be showed to be an isomorphism, thus yielding the main theorem.

Let $X \in Sh(C) \subset SSh(C)$ be a sheaf of sets that we regard as a simplicial sheaf where all sets of vertices of non-zero degree are empty and let $M \in Sh_{Ab}(C)$ be a sheaf of abelian groups.

2.1. **Definition of $H^n(X, M)$ as $Ext^n_{Sh_{Ab}(C)}(\mathbb{Z}[X]; M)$.** We define $H^n(X, M) := Ext^n_{Sh_{Ab}(C)}(\mathbb{Z}[X]; M)$. For this, we take an injectice resolution

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots \tag{2.1}$$

(a long exact sequence with injective objects I_n) and apply $Hom_{Sh_{Ab}(C)}(\mathbb{Z}, -)$ to get the chain complex

(2.2)

$$0 \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_0) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_1) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_2) \to \cdots .$$

Now the n-th homology group of X in M is the n-th homology of this complex, i.e. the quotient of the kernel of

$$Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_n) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_{n+1}) \tag{2.3}$$

by the image of

$$Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_{n-1}) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_n). \tag{2.4}$$

Notice that for 2.1 there can exist multiple different resolutions, however they will give rise to the same *Ext*-groups.

2.2. **Identifying Elements of** $[X, K(M, n)]$ **with Morphisms** $X \to \mathcal{K}_{(M,n)}$**.** The model structure axioms imply that any object can be embedded into a fibrant object with the embedding being a weak equivalence: We take the map from the object to the final object $*$ and use M2c to get a factorization into a cofibration, i.e. in our case a monomorphism, which is a weak equivalence and a fibration. In the case of $K(M, n)$ we obtain such a fibrant object and call it $\mathcal{K}_{(M,n)}$.

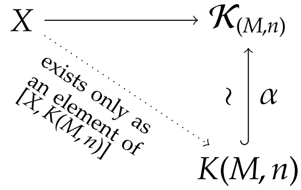
$$\begin{array}{c} \mathcal{K}_{(M,n)} \text{ fibrant} \\ \wr \Big\uparrow \alpha \\ K(M,n) \end{array} \tag{2.5}$$

Note that while we have a clear picture of $K(M, n)$ by construction, we dont know much about the structure of $\mathcal{K}_{(M,n)}$.

The weak equivalence between $K(M, n)$ and $\mathcal{K}_{(M,n)}$ translates to an isomorphism in the homotopy category, and we have

$$[X, \mathcal{K}_{(M,n)}] = [X, K(M, n)]. \tag{2.6}$$

As $\mathcal{K}_{(M,n)}$ is fibrant (and like all objects fibrant), the isomorphism in theorem 1 in [Quillen, 1967, p.1.3] yields $[X, \mathcal{K}_{(M,n)}] = \pi(X, \mathcal{K}_{(M,n)})]$. This means that a morphism in the homotopy category with codomain $\mathcal{K}_{(M,n)}$, i.e. an element of $[X, \mathcal{K}_{(M,n)}]$, can be lifted to an actual morphism $X \to \mathcal{K}_{(M,n)}$. Thus we can associate to elements of $[X, K(M, n)]$ morphisms $X \to \mathcal{K}_{(M,n)}$.



(2.7)

2.3. **The Corresponding Chain Complexes to X and $K(M,n)$.** If we take a closer look on the sheaves of chain complexes associated with $\mathbb{Z}[X]$ and $K(M,n)$ via Dold-Kan, we see that they are of the form

$$N\mathbb{Z}[X] = N\mathbb{Z}[X] \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots \tag{2.8}$$

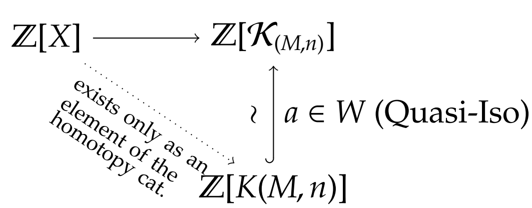
and

$$N\mathbb{Z}[K(M,n)] = 0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow M \leftarrow M' \leftarrow \cdots \tag{2.9}$$

with M at the n-th position, 0 for $i < n$ and some entries for $i > n$ that aren't of interest for us right now. This especially means that there won't be any nontrivial morphism from X to $K(M,n)$ for $n > 0$, i.e. the dotted arrow in 2.7 really only makes sense in the homotopy category. We have the projection p discarding all terms with $i > n$:

$$\begin{array}{ccccccccccc} 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & M & \longleftarrow & M' & \longleftarrow & \cdots \\ &&&&&&&& \Big\downarrow{\scriptstyle p} &&&& \\ 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & M & \longleftarrow & 0 & \longleftarrow & \cdots \end{array} \tag{2.10}$$

2.4. **Switching All to the Category of Chain Complexes of Sheaves of Abelian Groups.** Now we apply the free abelian group functor and the diagram in 2.7 becomes



(2.11) .

By the means of Dold-Kan correspondence, we regard that diagram as one in the category of chain complexes. While the chain complex structure of $\mathbb{Z}[\mathcal{K}_{(M,n)}]$ is complicated (and won't be important for us) and the one of $\mathbb{Z}[K(M,n)]$ is as described above, $\mathbb{Z}[X]$ is, as a result of the trivial simplicial structure of X, concentrated at degree 0, i.e. the complex is

$$N\mathbb{Z}[X] = \mathbb{Z}[X] \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots \tag{2.12}$$

With the map p we get

$$\begin{array}{ccccc} N\mathbb{Z}[X] & \longrightarrow & N\mathbb{Z}[\mathcal{K}_{(M,n)}]_* & & \\ & \searrow & \wr\uparrow a & & \\ & & N\mathbb{Z}[K(M,n)] & \xrightarrow{p} & (\cdots \leftarrow 0 \leftarrow M \leftarrow 0 \leftarrow \cdots) \end{array} . \tag{2.13}$$

2.5. **Getting Maps into an Injective Resolution of M.** Now we form the pushout on the right hand side which gives us a complex $\mathcal{E}_*$ that is unique up to unique isomorphy:

$$\begin{array}{ccccc} N\mathbb{Z}[X] \longrightarrow & N\mathbb{Z}[\mathcal{K}_{(M,n)}] & \longrightarrow & \mathcal{E}_* \\ & \wr\uparrow a & & \wr\uparrow a' \\ & N\mathbb{Z}[K(M,n)] & \xrightarrow{p} & (\cdots \leftarrow 0 \leftarrow M \leftarrow 0 \leftarrow \cdots) \end{array} . \tag{2.14}$$

Note that in 2.14, the pushout map from $M[n]$ into $\mathcal{E}_*$ is a weak equivalence since the model structure guarantees that for pushouts of weak equivalences, as well as an injection because of lemma 2.

For what follows, we switch from left bounded chain complexes to unbounded ones, i.e. we use the forgetful embedding $Sh_{ChAb_+(C)} \hookrightarrow Sh_{ChAb(C)}$.

Now let

$$I_* = I_{n+1} \leftarrow I_n \leftarrow I_{n-1} \leftarrow \cdots \leftarrow I_1 \leftarrow I_0 \leftarrow 0 \leftarrow \cdots \tag{2.15}$$

where the I_i form an injective resolution of M. As all entries are injective, I_* is an injective object in the category of chain complexes. The map from M into I_0 gives us the following chain complex injection:

$$\begin{array}{ccccccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & M & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & I_{n+1} & \longleftarrow & I_n & \longleftarrow & I_{n-1} & \longleftarrow & \cdots & \longleftarrow & I_1 & \longleftarrow & I_0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array} \tag{2.16}$$

From the injectivity of I_* we obtain the dashed arrow

$$\begin{array}{ccccccc}
N\mathbb{Z}[X] & \longrightarrow & N\mathbb{Z}[\mathcal{K}_{(M,n)}] & \longrightarrow & \mathcal{E}* & \overset{\exists}{\dashrightarrow} & I_* \\
 & & \wr \uparrow a & & \wr \uparrow a' & \text{unique up to homotopy} & \\
 & & N\mathbb{Z}[K(M,n)] & \overset{p}{\twoheadrightarrow} & (\cdots \leftarrow 0 \leftarrow M \leftarrow 0 \leftarrow \cdots) & \hookrightarrow & I_*
\end{array} \tag{2.17}$$

As a' is also a weak equivalence, from homological algebra we get that the induced map is unique up to homotopy.

2.6. **Turning this Map into an Actual Element of $H^n(X, M)$.** Now the composition of the upper arrows in 2.14 gives a chain complex map from $\mathbb{Z}[X]$ to I_* that, written out as a chain complex map is

$$\begin{array}{ccccccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}[X] & \longleftarrow & 0 & \longleftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow f & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & I_{n+1} & \underset{\partial_{n+1}}{\longleftarrow} & I_n & \underset{\partial_n}{\longleftarrow} & I_{n-1} & \longleftarrow & \cdots & \longleftarrow & I_1 & \underset{\partial_1}{\longleftarrow} & I_0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array} \tag{2.18}$$

That means we have constructed a map $f : \mathbb{Z}[X] \to I_n$ that up to homotopy only depends on the original map $X \to \mathcal{K}_{(M,n)}$.

To have that this gives an element of $H^n(X, M) = Ext^n_{Sh_{Ab}(C)}(\mathbb{Z}[X]; M)$, we need to show that f is in the kernel of

$$\partial_{n+1} \circ - : Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_n) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_{n+1}) \tag{2.19}$$

and that homotopy only changes f by something in the image of

$$\partial_n \circ - : Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_{n-1}) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_n). \tag{2.20}$$

The former comes from the fact that 2.18 is a morphism of chain complexes and the commutativity the square to the left of f reads $\partial_{n+1} \circ f = 0$.

The latter is due to the observation that homotopy transfers to homotopy of chain complexes in 2.18; if f and g are homotopic maps, we get the diagram

(2.21)

$$\begin{array}{ccccccccccccccccc}
\cdots & \longleftarrow & 0 & \leftarrow & \mathbb{Z}[X]^2 & \leftarrow & \mathbb{Z}[X] & \leftarrow & \cdots & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & f-g & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \leftarrow & I_{n+1} & \underset{\partial_{n+1}}{\longleftarrow} & I_n & \underset{\partial_n}{\longleftarrow} & I_{n-1} & \leftarrow & \cdots & \longleftarrow & I_1 & \underset{\partial_1}{\longleftarrow} & I_0 & \longleftarrow & 0 & \longleftarrow & \cdots
\end{array}.$$

Hence $f - g$ is in the image of $\partial_n \circ$.

Thus we have defined a well defined assignment ϕ of elements of $[X, K(M, n)]$. A closer look at the construction and applying it appropriately to maps shows that ϕ is in fact natural in both X and M.

3. The Theorem $[X, K(M, n)] \cong H^n(X, M)$

Let X be a sheaf of sets and M be a sheaf of abelian groups on the small site $\mathcal{C}$.

Theorem 1. *For $n \geq 0$, the map $\phi : [X, K(M, n)] \to H^n(X, M)$ that was constructed in the previous section, is an isomorphism.*

4. The Proof

We will use an induction argument and start with the case $n = 0$:

Lemma 3. *The theorem holds in the case $n = 0$.*

Proof. For $n = 0$, the Eilenberg-MacLane object $K(M, n)$ is just the simplicial sheaf with locally M as the set of 0-vertices and no higher vertices, i.e. it is a set of descrete points. Hence it is fibrant and we can simply take $\mathcal{K}_{(M,n)} := K(M, n)$ in 2.5. Thus, the pushout a' of $a = Id_{K(M,n)}$ in 2.14 along p is again the identity $Id_{M[n]}$. Hence in this case $n = 0$; the map ϕ is simply postcomposition with the injective embedding $\partial_0 : M \to I_0$ or in other words the maps $f : \mathbb{Z}[X] \to I_0$ that we get out of diagram 2.17 are all maps from $\mathbb{Z}[X]$ to I_0 that factor through $\partial_0 : M \to I_0$. On the other hand, the elements of $H^0(X, M)$ are all elements of the kernel of D_1 in

$$0 \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_0) \xrightarrow{D_1} Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_1) \to Hom_{Sh_{Ab}(C)}(\mathbb{Z}[X], I_2) \to \cdots . \tag{4.1}$$

But D_1 is just the postcomposition with the map ∂_1 in

$$0 \to M \xrightarrow{\partial_0} I_0 \xrightarrow{\partial_1} I_1 \to I_2 \to \cdots \tag{4.2}$$

That together means for maps from $\mathbb{Z}[X]$ to I_0 that

$$f \in H^0(X, M) \Leftrightarrow \partial_1 \circ f = 0 \tag{4.3}$$

As sequence 4.2 is exact, ∂_0 is the kernel (in the category theoretical sense) of ∂_1, i.e. $\partial_1 \circ f = 0$ implies that f factors through ∂_0. Thus we have shown that each element of $H^0(X, M)$ can be obtained via ϕ, i.e. ϕ is surjective.

The injectivity in this case easily follows from the fact that if two maps $X \to M$ are mapped to the same element of $H^0(X, M)$ by post-composition with ∂_0, they must be equal because ∂_0 is injective.

□

Now we consider the situation of having a short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0 \tag{4.4}$$

of sheaves of abelian groups.

If we take injective resolutions I_*, J_* and K_* for the 3 sheaves of abelian groups, we get the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I_1 & \longrightarrow & J_1 & \longrightarrow & K_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & I_2 & \longrightarrow & J_2 & \longrightarrow & K_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots
\end{array}
\tag{4.5}
$$

which is commutative with exact lines and columns where the horizontal maps are consecutively obtained by using the universal property of the injective objects. We then calculate the homology groups for X in our 3 sheaves and making use of the snake lemma multiple times on the resulting grid diagram, we get a long exact sequence in homology

(4.6)
$$\cdots \to H^{n-1}(X, M_3) \to H^n(X, M_1) \to H^n(X, M_2) \to H^n(X, M_3) \to H^{n+1}(X, M_1) \to \cdots$$

with zeroes for $n < 0$.

The short exact sequence 4.4 makes that the Eilenberg-MacLane spaces of the M_i form a fibre sequence

$$K(M_1, n) \to K(M_2, n) \to K(M_3, n) \tag{4.7}$$

for every n. This again extends (after Q, 3.5) to an exact sequence

(4.8)
$$\cdots \to \Omega(K(M_2, n)) \to \Omega(K(M_3, n)) \to K(M_1, n) \to K(M_2, n) \to K(M_3, n)$$

where any three consecutive terms form a fibre sequence. We use the fact that the construction of loop spaces produces for Eilenberg-MacLane objects the identity

$$\Omega(K(M,n)) = K(M,n-1) \tag{4.9}$$

and the arrows are the same as in the fibre sequence of degree $n-1$. Thus, with everywhere chosing n big enough, we get a long exact sequence

(4.10)

$$\cdots \to K(M_3, n-1) \to K(M_1, n) \to K(M_2, n) \to K(M_3, n) \to K(M_1, n+1) \to \cdots$$

which is a fibre sequence at every term. Applying lemma 1 gives us the long exact sequence

(4.11)

$$\cdots \to [X, K(M_3, n-1)] \to [X, K(M_1, n)] \to [X, K(M_2, n)] \to [X, K(M_3, n)] \to \cdots .$$

With the map ϕ from above, we get the following diagram:

(4.12)

$$\begin{array}{ccccccccc}
\cdots \to [X, K(M_3, n-1)] & \longrightarrow & [X, K(M_1, n)] & \longrightarrow & [X, K(M_2, n)] & \longrightarrow & [X, K(M_3, n)] & \longrightarrow & \cdots \\
\downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \\
\cdots \longrightarrow H^{n-1}(X, M_3) & \longrightarrow & H^n(X, M_1) & \longrightarrow & H^n(X, M_2) & \longrightarrow & H^n(X, M_3) & \longrightarrow & \cdots
\end{array}$$

Lemma 4. *This diagram 4.12 is commutative.*

Proof. This can be calculated by explicitly using injective resolutions and getting maps between them in a similar manner as for diagram 4.5 and checking that the construction of ϕ is compatible with both horizontal maps using the degree of freedom that the construction steps that are only up to homotopy give. □

With the goal of using injective objects to show the theorem with natural induction, we show the following lemmas.

Lemma 5. *If I is injective, then $K(I, n)$ is fibrant.*

Proof. By definition of what being fibrant means, we simply have to show that for every diagram

$$\begin{CD} A @>f>> K(I,n) \\ @V{\wr}V{i}V @VVV \\ B @>>> * \end{CD} \tag{4.13}$$

with A and B arbitrary simplicial sheaves and i a trivial cofibration, a lifting k such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & K(I,n) \\ {\wr}\big\downarrow i & \overset{k}{\nearrow} & \big\downarrow \\ B & \xrightarrow{\qquad} & ** \end{array} \tag{4.14}$$

commutes does exist.

But via Dold-Kan correspondence and construction of $K(I,n)$ as just corresponding to $I[n] = (\cdots \leftarrow 0 \leftarrow I \leftarrow 0 \leftarrow \cdots)$, this translates to the diagram

$$\begin{array}{ccc} N\mathbb{Z}[A] & \xrightarrow{\quad f. \quad} & I[n] \\ {\wr}\big\downarrow i. & \overset{k.}{\nearrow} & \big\downarrow \\ N\mathbb{Z}[B] & \xrightarrow{\qquad} & 0 \end{array} \tag{4.15}$$

where the preservation of the model structure under Dold-Kan ensures that $i.$ is a cofibration, i.e. a monomorphism in the model category of chain complexes of sheaves of abelian groups, that is a mono in every degree.

As $I[n]$ is trivial in any degree different from n, so is $f.$ and can be $k.$. Hence, as the cokernel of $i.$ is acyclic, we can approach the problem pointwise and all we need is a lift k_n in

$$\begin{array}{ccc} N\mathbb{Z}[A]_n & \xrightarrow{\quad f_n \quad} & I \\ {\scriptstyle i_n}\big\downarrow & \overset{k_n}{\nearrow} & \big\downarrow \\ N\mathbb{Z}[B]_n & \longrightarrow & 0 \end{array} . \tag{4.16}$$

This is exactly what the injectivity of I gives us by definition.

Hence we have constructed a lifting $k.$ of the chain complex maps which again gives us the desired map k. □

Lemma 6. *If I is injective and $n > 0$, then* $[X, K(I, n)] = 0$

Proof. From lemma 5 we know that $K(I, n)$ is fibrant.

That means that every morphism $F \in [X, K(M, n)]$ in the homotopy category can be lifted to a morphism $f : X \to K(M, n)$ in the actual sheaf category (choose $\mathcal{K}_{(M,n)} = K(M, n)$ in 2.5).

But as stated after regarding $K(M, n)$ as a chain complex in 2.9, there are no nontrivial morphisms between X, whose complex is concentrated in degree 0, and $K(M, n)$ with a zero at degree $0 < n$.

□

Lemma 7. *If I is injective and $n > 0$, then* $H^n(X, I) = 0$

Proof. Calculating $H^n(X, I) = Ext^n_{Sh_{Ab}(C)}(\mathbb{Z}[X], I)$ can be done with any injective resolution of I. We choose the resolution

$$0 \to I \xrightarrow{Id} I \xrightarrow{Id} I \xrightarrow{Id} I \xrightarrow{Id} \cdots . \tag{4.17}$$

Of course, the functor $Hom(\mathbb{Z}[X], -)$ preserves the identity isomorphisms, giving the complex

(4.18)

$$0 \to Hom(\mathbb{Z}[X], I) \xrightarrow{Id} Hom(\mathbb{Z}[X], I) \xrightarrow{Id} Hom(\mathbb{Z}[X], I) \xrightarrow{Id} Hom(\mathbb{Z}[X], I) \xrightarrow{Id} \cdots .$$

with trivial homology for $n > 0$. □

As there are enough injectives, we can embed the sheaf of abelian groups M from the theorem into an injective object I. With the cokernel N of this embedding, we have the short exact sequence

$$0 \to M \hookrightarrow I \twoheadrightarrow N \to 0. \tag{4.19}$$

Replacing in 4.4 M_1 with M, M_2 with I and M_3 with N, the commutative diagram 4.12 reads

(4.20)

$$\begin{array}{ccccccccc} \cdots \longrightarrow & [X,K(I,n-1)] & \longrightarrow & [X,K(N,n-1)] & \longrightarrow & [X,K(M,n)] & \longrightarrow & [X,K(I,n)] & \longrightarrow \cdots \\ & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & \\ \cdots \longrightarrow & H^{n-1}(X,I) & \longrightarrow & H^{n-1}(X,N) & \longrightarrow & H^n(X,M) & \longrightarrow & H^n(X,I) & \longrightarrow \cdots \end{array}$$

Now we use induction for n:
As shown above, the theorem holds for $n = 0$.

Lemma 8. *The theorem holds in the case $n = 1$.*

Proof. The first 5 terms in 4.20 read

(4.21)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & [X,K(M,0)] & \longrightarrow & [X,K(I,0)] & \longrightarrow & [X,K(N,0)] & \longrightarrow & [X,K(M,1)] & \rightarrow & [X,K(I,1)] \\ \downarrow & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ 0 & \longrightarrow & H^0(X,M) & \longrightarrow & H^0(X,I) & \longrightarrow & H^0(X,N) & \longrightarrow & H^1(X,M) & \longrightarrow & H^1(X,I) \end{array}$$

From lemmas 6 and 7 we get that $[X,K(I,1)]$ and $H^1(X,I)$ are zero.
We also know from lemma 3 that the three vertical maps in degree $n = 0$ are isomorphisms.
Thus diagram 4.21 becomes

(4.22)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & [X,K(M,0)] & \longrightarrow & [X,K(I,0)] & \longrightarrow & [X,K(N,0)] & \longrightarrow & [X,K(M,1)] & \longrightarrow & 0 \\ \downarrow & & \cong\downarrow \phi & & \cong\downarrow \phi & & \cong\downarrow \phi & & \downarrow \phi & & \downarrow \phi \\ 0 & \longrightarrow & H^0(X,M) & \longrightarrow & H^0(X,I) & \longrightarrow & H^0(X,N) & \longrightarrow & H^1(X,M) & \longrightarrow & 0 \end{array}$$

and regarding the lines' exactness, an easy form of diagram chase or using the five-lemma forces $\phi : [X,K(M,1)] \to H^1(X,M)$ to also be an isomorphism. □

Thus for $n > 1$ we can assume that ϕ is an isomorphism for $n-1$ and the theorem is reduced to the following last lemma:

Lemma 9. *If $n > 1$ and the theorem's conclusion holds for $n-1$, then it also holds for n.*

Proof. We apply the assumption that the result is true for $n-1$ to the group sheaf N, obtaining that

$$\begin{array}{c} [X, K(N, n-1)] \\ \Big\downarrow \phi \\ H^{n-1}(X, N) \end{array} \tag{4.23}$$

is an isomorphism.

As $n > 1$, from Lemmas 6 and 7 we have that the terms with I are zero, i.e. the part of diagram 4.20 becomes

(4.24)

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & [X, K(N, n-1)] & \longrightarrow & [X, K(M, n)] & \longrightarrow & 0 \\ & & \Big\| = & & \Big\downarrow \cong & & \Big\downarrow \phi & & \Big\| = \\ \cdots & \longrightarrow & 0 & \longrightarrow & H^{n-1}(X, N) & \longrightarrow & H^{n}(X, M) & \longrightarrow & 0 \end{array}.$$

The exactness of the long exact sequences implies that the horizontal arrows are also isomorphisms. Hence in the commutative diagram

$$\begin{array}{ccc} [X, K(N, n-1)] & \xrightarrow{\cong} & [X, K(M, n)] \\ \Big\downarrow \cong & & \Big\downarrow \phi \\ H^{n-1}(X, N) & \xrightarrow{\cong} & H^{n}(X, M) \end{array} \tag{4.25}$$

the map ϕ must also be an isomorphism. □

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